THE UNITED REPUBLIC OF TANZANIA

MINISTRY OF EDUCATION, SCIENCE AND TECHNOLOGY

NATIONAL EXAMINATIONS COUNCIL OF TANZANIA

CANDIDATES' ITEM RESPONSE ANALYSIS REPORT ON THE ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION (ACSEE) 2025

142 ADVANCED MATHEMATICS

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FOREWORD

The National Examinations Council of Tanzania has prepared this analysis report based on candidates' responses to the Advanced Certificate of Secondary Education Examination (ACSEE) in 2025 Advanced Mathematics. The report highlights how candidates responded to the questions, identifying the reasons behind their strengths and weaknesses. It aims to provide feedback to all education stakeholders, which can be used to improve students' performance in Advanced Mathematics.

Overall, the candidates' performance in Advanced Mathematics was good, with 97.51 percent of candidates passing the examination. The analysis indicates that candidates performed well in 14 topics out of the 18 topics assessed. These topics included *Logic*, *Functions*, *Linear Programming*, *Coordinate Geometry*, *Statistics*, *Numerical Methods*, *Sets*, *Trigonometry*, *Algebra*, *Vectors*, *Calculating Devices*, *Probability*, *Hyperbolic Functions*, and *Differentiation*. This good performance was attributed to candidates' ability to identify the requirements of the questions, recall, and apply the appropriate formulae and rules.

The candidates' performance in the topics of *Differential Equations*, *Coordinate Geometry II*, and *Complex Numbers* was average, while their performance in *Integration* was weak. The poor performance in *Integration* was attributed to a lack of sufficient understanding of the concept of integration and its application, as well as misinterpretation of the question.

The National Examinations Council of Tanzania believes that the feedback provided in this analysis will be helpful to education stakeholders in developing effective strategies to enhance future performance in Advanced Mathematics.

The Council sincerely thanks all individuals who contributed to the development and completion of this report.

Prof. Said Ally Mohamed
EXECUTIVE SECRETARY

1.0 INTRODUCTION

The Advanced Mathematics papers in the ACSEE 2025 examination were set based on the Advanced Mathematics syllabus for Advanced Secondary Education and the examination format issued in 2010 and 2019, respectively.

The Advanced Mathematics examination consisted of two papers: 142/1 Advanced Mathematics 1 and 142/2 Advanced Mathematics 2. Paper 142/1 comprised ten (10) compulsory questions, each worth ten (10) marks. Paper 142/2 was divided into two sections; A and B. Section A consisted of four (4) compulsory questions, each worth fifteen (15) marks. Section B consisted of four (4) optional questions, each worth twenty (20) marks, from which candidates were required to attempt two (2) questions.

A total of 18,136 candidates sat for the Advanced Mathematics examination in ACSEE 2025, of whom 17,666 (97.51%) passed the examination. In ACSEE 2024, 97.70 percent of the candidates who sat for the examination passed. In comparison, the overall performance in ACSEE 2025 declined by 0.19 percent. Figure 1 illustrates the trend of candidates' performance by grades for ACSEE 2024 and 2025.

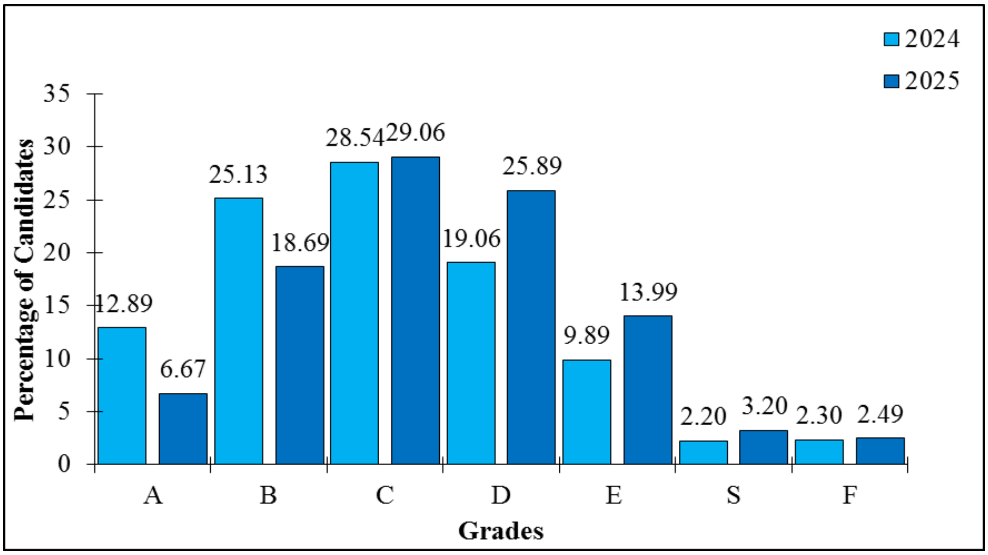


Figure 1: *The candidates' Performance on the ACSEE 2024 and 2025 in Advanced Mathematics*

Figure 1 shows that in ACSEE 2025, the percentage of candidates who scored grades A and B decreased, whereas the percentage of those who scored grades C, D, E, S, and F increased.

Section 2 of the report provides a detailed analysis of the candidates' performance and responses to each question. Section 3 outlines the candidates' performance across the examined topics, while Section 4 presents the conclusion and recommendations. This report also includes Appendix I, which shows the overall performance of the candidates by topic in ACSEE 2025, and Appendix II, which compares the candidates' performance in ACSEE 2024 and 2025.

2.0 ANALYSIS OF CANDIDATES' PERFORMANCE IN EACH QUESTION

This section presents an analysis of candidates' performance in the ACSEE 2025 Advanced Mathematics examination for each question. The analysis is based on three performance categories: 60–100%, 35–59%, and 0–34%, which indicate good, average, and weak performance, respectively.

2.1 142/1 ADVANCED MATHEMATICS 1

2.1.1 Question 1: Calculating Devices

The question examined the candidates' ability to use a non-programmable calculator to perform mathematical computations of different expressions. The question had two parts, (a) and (b), as follows:

(a) *Use a non-programmable calculator to evaluate the following expressions:*

(i) $\dfrac{\tan 25°30' - \sqrt[5]{0.03e^{-3}}}{\ln 3.2 + 0.006e^{0.3}}$ *correctly to six significant figures*

(ii) $\displaystyle\sum_{n=4}^{7} \frac{2^{-n}(n!)}{\ln(0.3n)}$ *correct to three decimal places.*

(b) *If* $A_t = A_5 + A_0 e^{-kt}$, $k = -0.1386$, $A_5 = 20$ *and* $A_{20} = 40$, *by using a non-programmable calculator, compute the value of* A_{16} *correct to 2 decimal places.*

The data indicates that out of 18,136 candidates, 6,054 (33.38%) scored from 0 to 3.0 marks, 173 (0.95%) scored from 3.5 to 5.5 marks, and 11,909 (65.67%) scored from 6.0 to 10 marks. The candidates' performance on this question was generally good, as 66.62 percent passed. Figure 2 summarises the overall performance of candidates with this question.

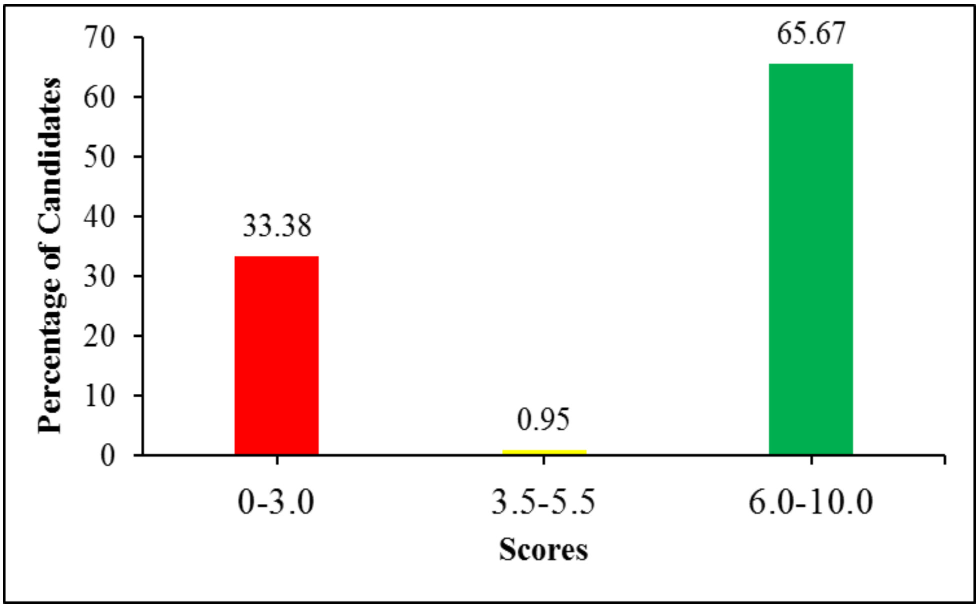


Figure 2: *The Candidates' Performance in Question 1 of Paper 1*

Analysis of responses shows that the candidates who performed well on this question had sufficient knowledge on the use of a non-programmable calculator. In part (a) (i), the candidates set the calculator in degree mode and fixed it to six significant figures. Then, they evaluated $\frac{\tan 25°30' - \sqrt[5]{0.03e^{-3}}}{\ln 3.2 + 0.006e^{0.3}}$ by correctly inserting the inputs into the calculator to obtain 0.174857. Similarly, in part (a) (ii), they fixed the calculator to three decimal places and correctly entered the inputs for $\sum_{n=4}^{7} \frac{2^{-n}(n!)}{\ln(0.3n)}$ into the calculator to obtain 89.686, which is the correct answer. In part (b), the candidates were able to enter the correct value of A_5, A_{20} and $t = 20$ into the calculator in order to get the value of A_0. They used the correct values of A_0, A_5 and $t = 16$ to get the correct answer $A_{16} = 31.49$. Extract 1.1

shows a sample of correct responses from a candidate who answered this question correctly.

1.	(a) i. 0.174857	
	ii. 89.686	
	(b) A16 = 31.49	

Extract 1.1: A sample of correct responses to question 1 of paper 1

In Extract 1.1, the candidate correctly used the non-programmable calculator to compute the values of the given expressions.

On the other hand, the candidates who had weak performance on this question faced different challenges. In part (a) (i), some candidates set the calculator in radian mode and failed to change the angle $25^{\circ}30'$ into radians. The mistake led to a wrong value of the expression, such as -0.34638. In part (a) (ii), some candidates wrongly entered the inputs of the expression $\sum_{n=4}^{7}\frac{2^{-n}(n!)}{\ln(0.3n)}$ resulting in wrong values such as 0.457. In part (b), the candidates failed to use the calculator to compute the correct value of A_0, and as a result, they were unable to determine the correct value of A_{16}. Extract 1.2 presents a sample of an incorrect response from a candidate who answered this question incorrectly.

1.	a. i. 0.096141	
	ii. 32.908	
	b. 1463469.91	

Extract 1.2: A sample of incorrect responses to question 1 of paper 1

In Extract 1.2, the candidate failed to use the calculator properly to compute the values of the given expressions.

2.1.2 Question 2: Hyperbolic Functions

The question examined the candidate's ability to prove hyperbolic identities using either definitions of hyperbolic functions or basic hyperbolic identities, and apply the techniques of differentiation to differentiate hyperbolic functions, apply the concept of integration to determine the area under the curve and the volume of the solid of revolution about the $x-$axis. The question stated as follows:

(a) Prove that $\frac{1+\tanh x}{1-\tanh x}=\cosh 2x+\sinh 2x$.

(b) Differentiate $\ln(\tanh x)$ *with respect to* x *and simplify your answer.*

(c) Find the area enclosed by the curve $y=\cosh x$, $x=\ln 2$, *the* $x-$*axis, and the* $y-$*axis; hence, find the volume obtained when this area is rotated completely about the* $x-$*axis.*

The data indicates that out of 18,136 candidates, 6,379 (35.17%) scored from 0 to 3.0 marks, 3,901 (21.51%) candidates scored from 3.5 to 5.5 marks, and 7,856 (43.32%) candidates scored from 6.0 to 10 marks. Figure 3 summarises the overall performance of candidates on this question.

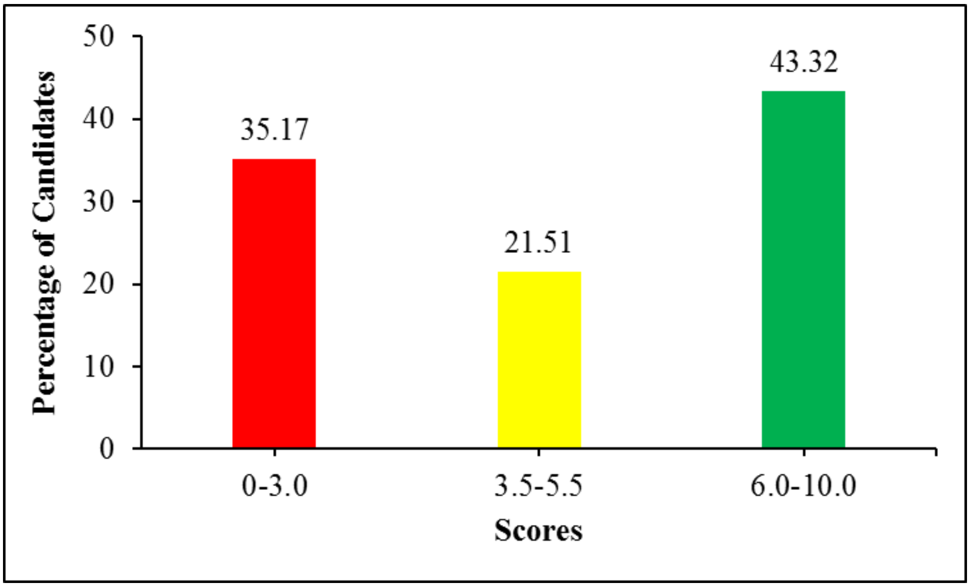


Figure 3: *The Candidates' Performance in Question 2 of Paper 1*

Figure 3 shows that the candidates' performance on this question was good, as 64.83 percent passed. The candidates who performed well demonstrated sufficient knowledge and competence in hyperbolic functions. In part (a), the candidates applied the definition of hyperbolic tangent, $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, to prove the given identity. They substituted this definition into $\frac{1+\tanh x}{1-\tanh x}$, and on simplifying, they ended up with $e^{2x} = \cosh 2x + \sinh 2x$. Others considered the left-hand side expression, $\cosh 2x + \sinh 2x$, and substituted the identities $\cosh 2x = \frac{1+\tanh^2 x}{1-\tanh^2 x}$ and $\sinh 2x = \frac{2\tanh x}{1-\tanh^2 x}$ to get $\frac{1+\tanh^2 x}{1-\tanh^2 x} + \frac{2\tanh x}{1-\tanh^2 x}$, which was simplified to $\frac{1+\tanh x}{1-\tanh x}$. In part (b), the candidates differentiated the given function $\ln(\tanh x)$ using the concept of the derivative of a composite function with the chain rule and obtained $\frac{\operatorname{sech}^2 x}{\tanh x}$. Then, they simplified and got $2\operatorname{cosech} 2x$. In part (c), the candidates sketched the curve $y = \cosh x$ in the $xy-$plane to identify the required region. With the help of the sketch, the candidates calculated the area under the curve by using a correct formula, $A = \int_a^b y dx$, such that $A = \int_0^{\ln 2} \cosh x dx$ and obtained $A = \frac{3}{4} = 0.75$ square units. Then, they calculated the volume of solid of revolution about the x-axis by using the formula, $V = \pi \int_a^b y^2 dx$ and then substituted $y = \cosh x$, $a = 0$, and $x = \ln 2$, resulting in $V = 2.5614$ cubic units. Extract 2.1 is a sample of correct responses from one of the candidates who answered the question correctly.

2. a/ Soln:

Considering the right hand side;

Using t- formulae:

$$\cosh 2x = \frac{1+t^2}{1+t} = \frac{1+t^2}{1-t^2}$$

$$\sinh 2x = \frac{2t}{1-t^2} \quad \text{where } t = \tanh x$$

then,

$$\cosh 2x + \sinh 2x = \frac{1+\tanh^2 x}{1-\tanh^2 x} + \frac{2\tanh x}{1-\tanh^2 x}$$

$$\cosh 2x + \sinh 2x = \frac{1+\tanh^2 x + 2\tanh x}{1-\tanh^2 x}$$

$$\cosh 2x + \sinh 2x = \frac{(1+\tanh x)^2}{1-\tanh^2 x}$$

but

$$1-\tanh^2 x = (1+\tanh x)(1-\tanh x)$$

$$\cosh 2x + \sinh 2x = \frac{(1+\tanh x)^2}{(1+\tanh x)(1-\tanh x)}$$

2 a/ $\cosh 2x + \sinh 2x = \dfrac{1+\tanh x}{1-\tanh x}$

hence proved.

b/ Soln.

Let $y = \ln(\tanh x)$

$$\frac{dy}{dx} = \frac{1}{\tanh x} \cdot \operatorname{sech}^2 x .$$

$$\frac{dy}{dx} = \frac{\cosh x}{\sinh x} \cdot \frac{1}{\cosh^2 x} .$$

$$\frac{dy}{dx} = \frac{1}{\sinh x \cosh x} .$$

but

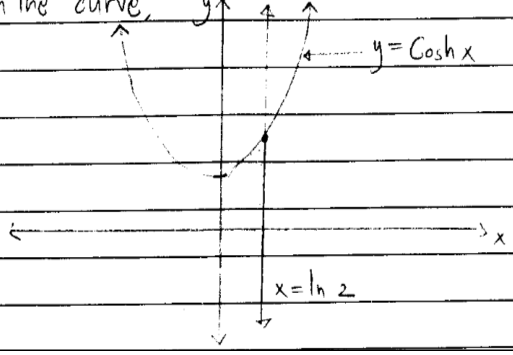
$$\sinh x \cosh x = \frac{\sinh 2x}{2}$$

$$\frac{dy}{dx} = \frac{2}{\sinh 2x} = 2\operatorname{cosech} 2x .$$

$$\therefore \frac{d(\ln(\tanh x))}{dx} = 2\operatorname{cosech} 2x .$$

c/ Soln.

From the curve;



2. c/ from $A = \int_{x_1}^{x_2} y \, dx$

$$A = \int_{0}^{\ln 2} \cosh x \, dx = \sinh x \Big|_{0}^{\ln 2}$$

$$A = \sinh(\ln 2) - \sinh(0)$$

$$A = \frac{3}{4} - 0$$

$$A = \frac{3}{4} \text{ square units}$$

then

$$V = \pi \int_{x_1}^{x_2} y^2 \, dx$$

$$V = \pi \int_{0}^{\ln 2} \cosh^2 x \, dx = \pi \int_{0}^{\ln 2} \frac{1}{2}(1 + \cosh 2x) \, dx$$

$$V = \frac{\pi}{2} \int_{0}^{\ln 2} dx + \frac{\pi}{2} \int_{0}^{\ln x} \cosh 2x \, dx$$

$$V = \frac{\pi}{2} \left(x + \frac{\sinh 2x}{2} \right) \Big|_{0}^{\ln 2}$$

$$V = \frac{\pi}{2} \left[\left(\ln 2 + \frac{\sinh 2(\ln 2)}{2} \right) - \left(0 + \frac{\sinh 2(0)}{2} \right) \right]$$

$$V = \frac{\pi}{2} \left(\ln 2 + \frac{15}{16} \right)$$

$V = 2.56$ ~~sq~~ cubic units

$\therefore$ Area enclosed, $A = \frac{3}{4}$ square units

Volume, $V = 2.56$ cubic units

Extract 2.1: A sample of correct responses to question 2 of paper 1

In Extract 2.1, in part (a), the candidate proved the hyperbolic identities using the definition of hyperbolic tangent, while in part (b), the candidate differentiated the given hyperbolic expression using the chain rule. In part (c), the candidate applied correct formulae to find the area under the curve and the volume of the solid of revolution about the x-axis.

However, some candidates performed poorly on this question due to several challenges. In part (a), they failed to recall the correct definition of the hyperbolic tangent; for example, they substituted $\tanh x = e^{2x} - 1$ and ended up getting incorrect answers. Others applied wrong identities, such as $\cosh^2 x + \sinh^2 x = 1$ instead of $\cosh^2 x - \sinh^2 x = 1$. In part (b), the candidates failed to apply the proper techniques of differentiation, such as writing $\frac{d}{dx}\left(\ln\left(\tanh x\right)\right) = \frac{1}{\tanh x}\ln \operatorname{sech}^2 x$. In part (c), the candidates used incorrect formulae in finding the area under the given curve and the volume of the solid of revolution about the x-axis. They used the formula $A = \int f(a)dx$ to find the area under the curve but did not define the variable a. In the case of volume, they applied an incorrect formula, $V = \pi\int_a^b xydx$, which resulted in wrong answers such as $V = 3.27$ cubic units. Furthermore, other candidates wrote $V = \int_a^b \pi y dx$. Extract 2.2 presents a sample of incorrect responses from a candidate who answered the question incorrectly.

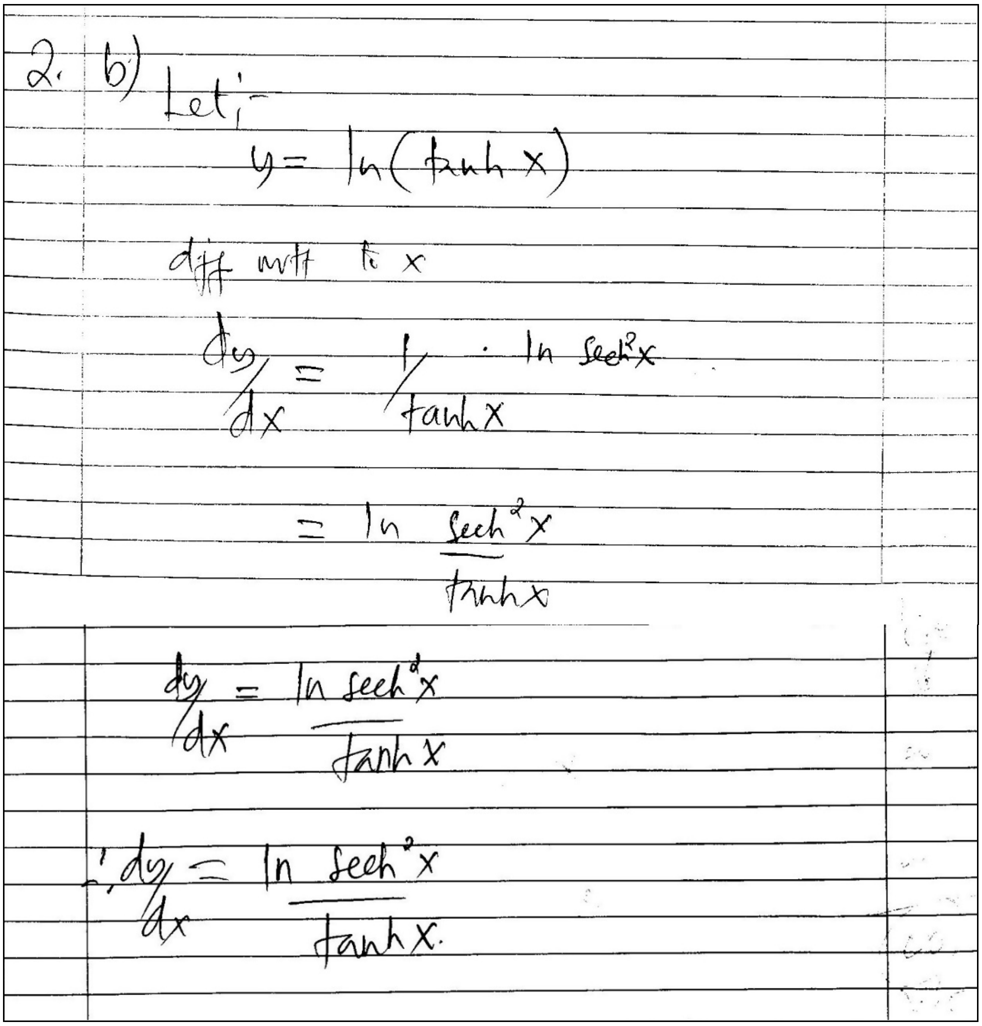
2. (a) Consider L.H.S

$$\frac{1+\tanh x}{1-\tanh x}$$

$$\frac{1+e^{2x}-1}{1-e^{2x}+1}$$

$$\cosh^2 x + \sinh^2 x = 1$$

$$\therefore \frac{1+\tanh x}{1-\tanh x} = \cosh 2x + \sinh 2x$$


2. b) Let;-

$y = \ln(\tanh x)$

diff wrt to x

$\frac{dy}{dx} = \frac{1}{\tanh x} \cdot \ln \operatorname{sech}^2 x$

$= \frac{\ln \operatorname{sech}^2 x}{\tanh x}$

$\frac{dy}{dx} = \frac{\ln \operatorname{sech}^2 x}{\tanh x}$

$\therefore \frac{dy}{dx} = \frac{\ln \operatorname{sech}^2 x}{\tanh x}$.

Extract 2.2: A sample of incorrect responses to question 2 of Paper 1

In Extract 2.2, part (a), the candidate applied an incorrect definition of hyperbolic function instead of $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ to prove the given identity. In part (b), the candidate applied chain rule incorrectly to differentiate the given expression.

2.1.3 Question 3: Linear Programming

The question assessed the candidates' ability to formulate and solve a given real-life linear programming problem, including writing linear equalities/inequalities and the objective function, and solving it graphically. The question stated that:

Rehema has 900 tonnes and 600 tonnes of bricks at Mtakuja and Tupendane villages, espectively. She has planned to build new houses at sites A, B, and C. She expects to use 500 tonnes of bricks at site A, 600 tonnes of bricks at site B, and 400 tonnes of bricks at site C. The transport cost in Tsh. is proportional to the distance covered in kilometres, as shown in the following table:

From/To	***A***	***B***	***C***
Mtakuja	*600*	*300*	*400*
Tupendane	*400*	*200*	*600*

If x and y are the number of bricks to be transported from Mtakuja to sites A and B, respectively,

(a) formulate the inequalities and objective function to be satisfied by x and y.

(b) find the number of bricks to be transferred from Mtakuja and Tupendane villages to each site.

The data indicates that out of 18,136 candidates, 1,657 (9.14%) scored from 0 to 3.0 marks and 16,479 (90.86%) candidates scored from 3.5 to 10 marks. This indicates that the general performance of the candidates was good. Figure 4 provides a summary of the candidates' performance on this question.

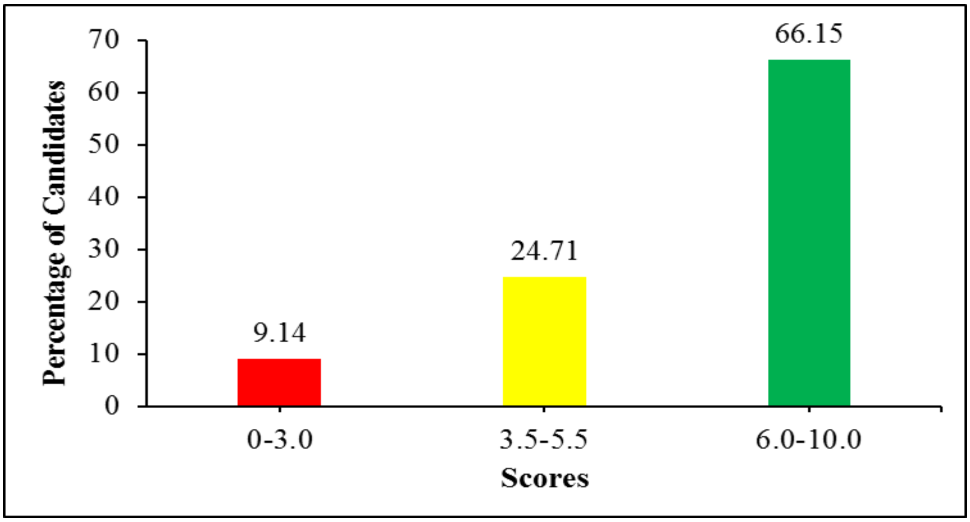
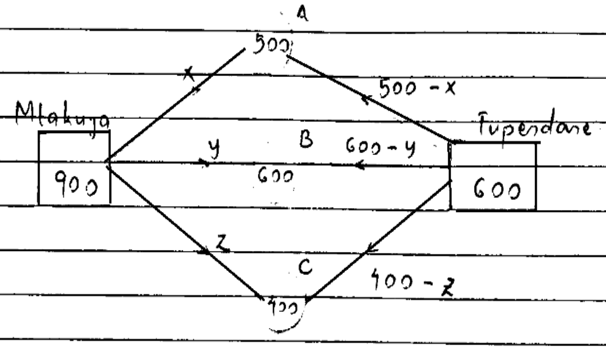


Figure 4: *The Candidates' Performance in Question 3 of Paper 1*

The data analysis reveals that 4759 (26.24%) candidates scored full marks. The candidates who performed well on this question had acquired the necessary competence regarding the concept of transportation problems. In part (a), by using the given decision variables, the candidates formulated the linear programming problem by writing the correct inequalities $x+y\leq 900$, $x+y\geq 500$, $x\leq 500$, $y\leq 600$, $x, y\geq 0$ and the objective function $Min\ f(x, y)=400x+300y+380000$. In part (b), the candidates plotted the inequalities on the xy-plane and identified the feasible region. Thereafter, they determined the corner points and substituted them into the objective function to obtain the minimum cost of Tsh 530,000 at $(0, 500)$. Extract 3.1 shows a sample of correct responses from a candidate who answered this question correctly.

3.



Here $x + y + z = 900$

$z = 900 - (x + y)$

Constraints:

a) $x \leq 500$ ——— i

$y \leq 600$ ——— ii

$z \leq 400$

$900 - (x + y) \leq 400$

$x + y \geq 500$ ——— iii

$z \geq 0$

$900 - (x + y) \geq 0$

$x + y \leq 900$ ——— iii

$x \geq 0$ ——— v

$y \geq 0$ ——— vi

Objective function:

$600x + 300y + 400z + 400(500 - x) + 200(600 - y) + 600(400 - z)$

$600x + 300y + 400z + 200000 - 400x + 120000 - 200y + 240000 - 600z$

3. $200x + 100y - 200z + 560{,}000$

but: $z = 900 - (x + y)$

$200x + 100y - 200[900 - (x + y)] + 560{,}000$

$200x + 100y - 180{,}000 + 200x + 200y + 560{,}000$

$400x + 300y + 380{,}000$

∴ Objective function

Minimize

$f(x, y) = 400x + 300y + 380{,}000$

Corner points	Values
A (0, 500)	530,000
B (0, 600)	560,000
C (300, 600)	680,000
D (500, 400)	700,000
E (500, 0)	580,000

Optimal solution: (0, 500)

Optimal value 530,000

3 ∴ 0 bricks from Mtakuja to site A

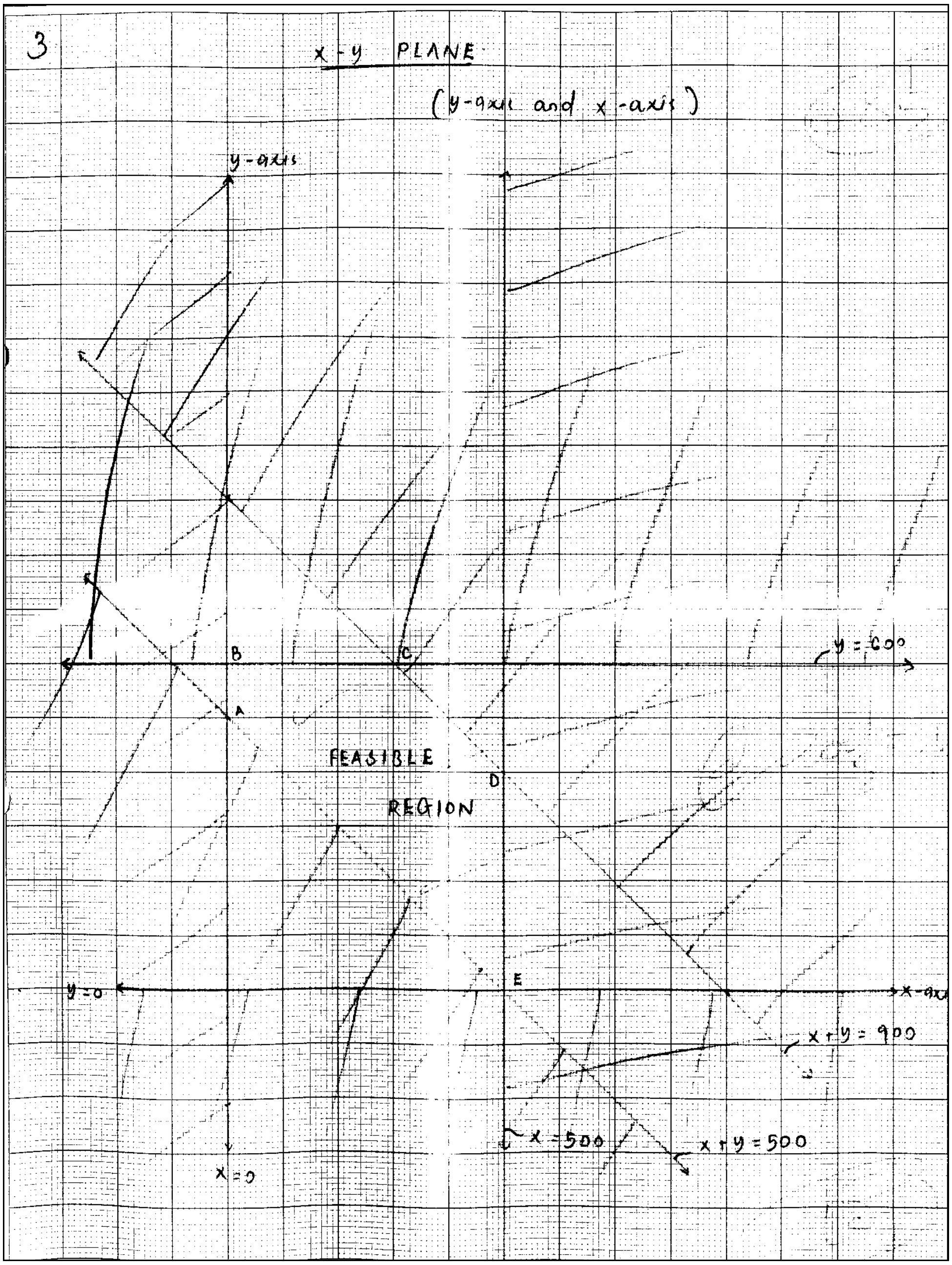
500 bricks from Mtakuja to site B

400 bricks from Mtakuja to site C

500 bricks from Tupendane to site A

100 bricks from Tupendane to site B

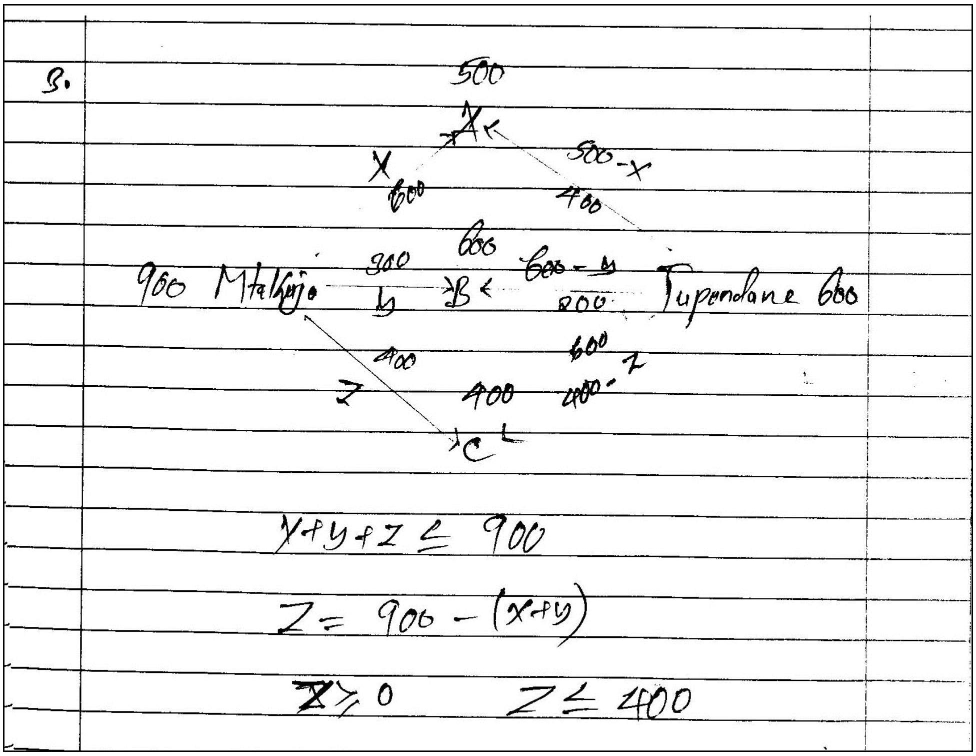
0 bricks from Tupendane to site C

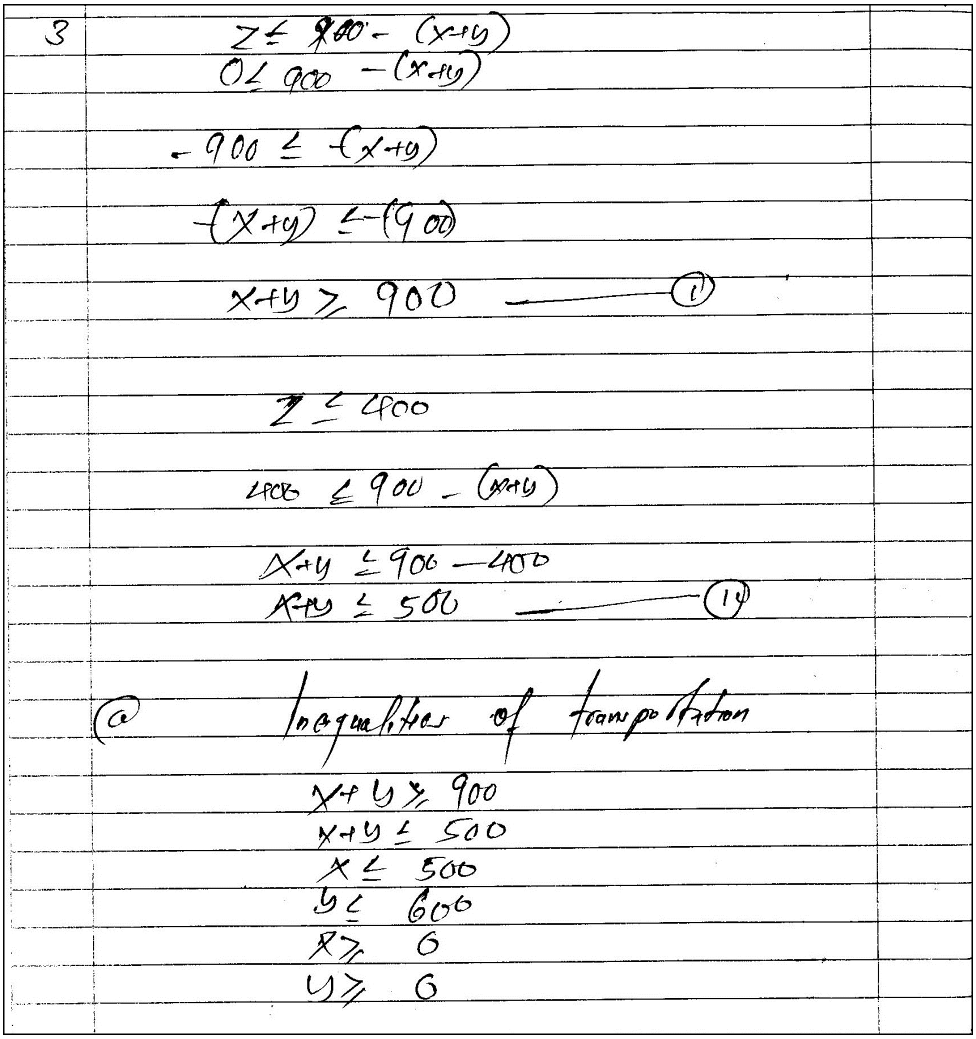


Extract 3.1: A sample of correct responses to question 3 of paper 1

In Extract 3.1, part (a), the candidate correctly formulated the constraints and the objective function. In part (b), he/she was able to identify the required feasible region and provided the correct number of tonnes of bricks to be transported to the sites.

Despite the outstanding performance of most candidates, some faced challenges with this question due to inadequate knowledge of formulating linear inequalities related to transportation problems. For example, in part (a), some candidates wrote incorrect constraints: $x+y \geq 900$, $x+y \leq 500$ $x \leq 500$, and $y \leq 600$. In part (b), they failed to draw the graph correctly, which led to an incorrect solution. Extract 3.2 shows a sample of an incorrect response from one of the candidates who answered the question incorrectly.





3 $z \leq 900 - (x+y)$

$0 \leq 900 - (x+y)$

$-900 \leq -(x+y)$

$-(x+y) \leq -(900)$

$x+y \geq 900$ ——— (i)

$z \leq 400$

$400 \leq 900 - (x+y)$

$x+y \leq 900 - 400$

$x+y \leq 500$ ——— (ii)

(a) Inequalities of transportation

$x+y \geq 900$

$x+y \leq 500$

$x \leq 500$

$y \leq 600$

$x \geq 0$

$y \geq 0$

Extract 3.2: A sample of incorrect responses to question 3 of paper 1

In Extract 3.2, the candidate failed to formulate the correct inequalities due to confusion between the signs $\leq$ and $\geq$. He/she also wrote an inequality $x + y + z \leq 900$ instead of an equality $x + y + z = 900$.

2.1.4 Question 4: Statistics

The question examined the candidates' ability to calculate measures of central tendency and dispersion from a frequency distribution table. The candidates were given the following table showing the frequency distribution of resistance values (in ohms) for 48 resistors:

Resistors	20.5-20.9	21.0-21.4	21.5-21.9	22.0-22.4	22.5-22.9	23.0-23.4
Frequency	3	10	11	13	9	2

Then, the candidates were required to:

(a) Use this information to find the mode, median, and 45^{th} percentile correct to 2 decimal places.

(b) Use the coding method to find the mean and standard deviation for the distribution correct to 4 significant figures.

This question was attempted by 18,136 (100%) candidates, of whom 15,384 (84.83%) scored 3.5 to 10 marks. Therefore, the performance of candidates on this question was generally good. Figure 5 summarises the candidates' performance on this question.

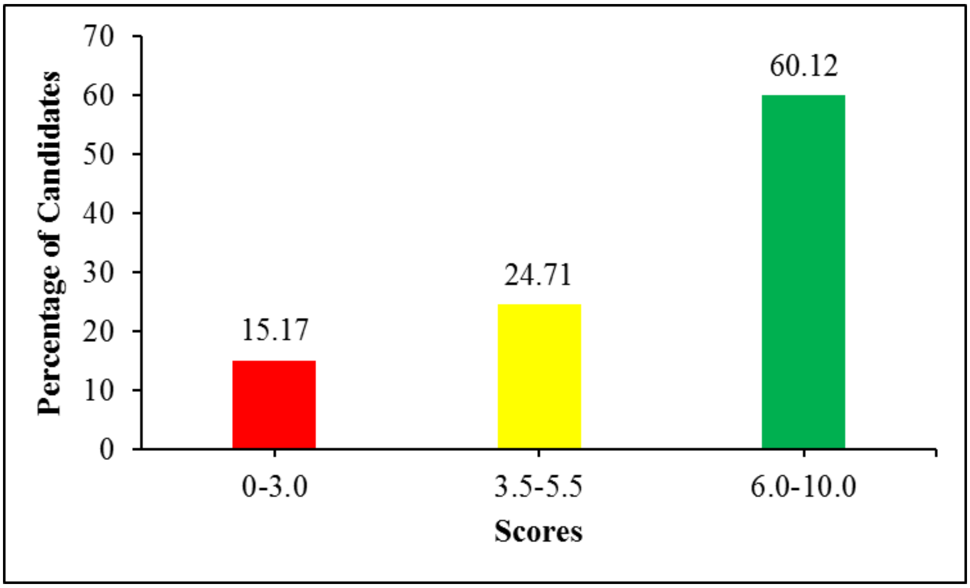


Figure 5: *The Candidates' Performance in Question 4 of Paper 1*

The data analysis shows that 2,151 (11.86%) candidates scored full marks. This indicates that they had gained sufficient competence on advanced tenets of statistics. In part (a), the candidates used the information from the given frequency distribution table and applied appropriate formulae to find mode $M = L + \left(\frac{t_1}{t_1 + t_2}\right)c$ and then substituted the values into the formula to obtain the correct answer $M = 22.12$. For the median, they used the formula $Median = L + \left(\frac{\frac{N}{2} - n_b}{n_w} \right) c$, and upon substituting the values, they obtained the $Median = 21.95$. In finding the 45^{th} percentile, the candidates applied the formula $P_{45} = L + \left(\frac{\frac{45N}{100} - n_b}{n_w} \right) c$ and then substituted the values to obtain 21.84, which is the correct answer. In part (b), the candidates prepared a new frequency distribution table containing the necessary values and applied correct formulae to find the mean and standard deviation using the coding method correct to 4 significant figures. They calculated the mean using the correct formula $\overline{X} = A + c\frac{\sum fu}{N}$ and substituted the values from the table to obtain $\overline{X} = 21.92$. In calculating the standard deviation, they applied the formula $\sigma = c\sqrt{\frac{\sum f_i u_i^2}{N} - \left(\frac{\sum f_i u_i}{N} \right)^2}$ and got $\sigma = 0.6447$.

Extract 4.1 shows a sample of a correct response from a candidate who answered this question correctly.

4. Consider the table below: A = 21.7

Interval	x	f	$u = \frac{x-A}{c}$	fu	u^2	fu^2
20.5 - 20.9	20.7	3	-2	-6	4	12
21.0 - 21.4	21.2	10	-1	-10	1	30-20=10
21.5 - 21.9	21.7	11	0	0	0	0
22.0 - 22.4	22.2	13	1	13	1	13
22.5 - 22.9	22.7	9	2	18	4	36
23.0 - 23.4	23.2	2	3	6	9	18
		$\Sigma = 48$		$\Sigma fu = 21$		$\Sigma fu^2 = 109 - 20 = 89$

a) from:

$$Mode = L + \left(\frac{t_1}{t_1 + t_2}\right)c$$

Modal class (22.0 - 22.4)

$L = 22.0 - 0.05$

$L = 21.95$

$$Mode = 21.95 + \left(\frac{2}{2+4}\right)0.5$$

Mode = 22.117

$\therefore$ Mode = 22.12 ohms

$$Median = L + \left(\frac{N/2 - Ol}{fw}\right)c$$

Median class = (21.5 - 21.9)

$L = 21.5 - 0.05$

$L = 21.45$

4. Median $= 21.45 + \left(\frac{24 - 13}{11}\right) 0.5$

$= 21.95$

$\therefore$ Median $= 21.95$ ohms

45th percentile $= L + \left(\frac{\frac{45}{100}N - \Sigma f_b}{f_w}\right) c$

$L = 21.45$

45th percentile $= 21.45 + \left(\frac{21.6 - 13}{11}\right) 0.5$

$= 21.84$

$\therefore$ 45th percentile is 21.84 ohms.

b) from Mean $= A + c\frac{\Sigma fu}{N}$

$\bar{x} = 21.7 + \frac{0.5 \times 21}{48}$

$\bar{x} = 2.192 \times 10^1$

$\therefore$ Mean $= 2.192 \times 10^1$ ohms.

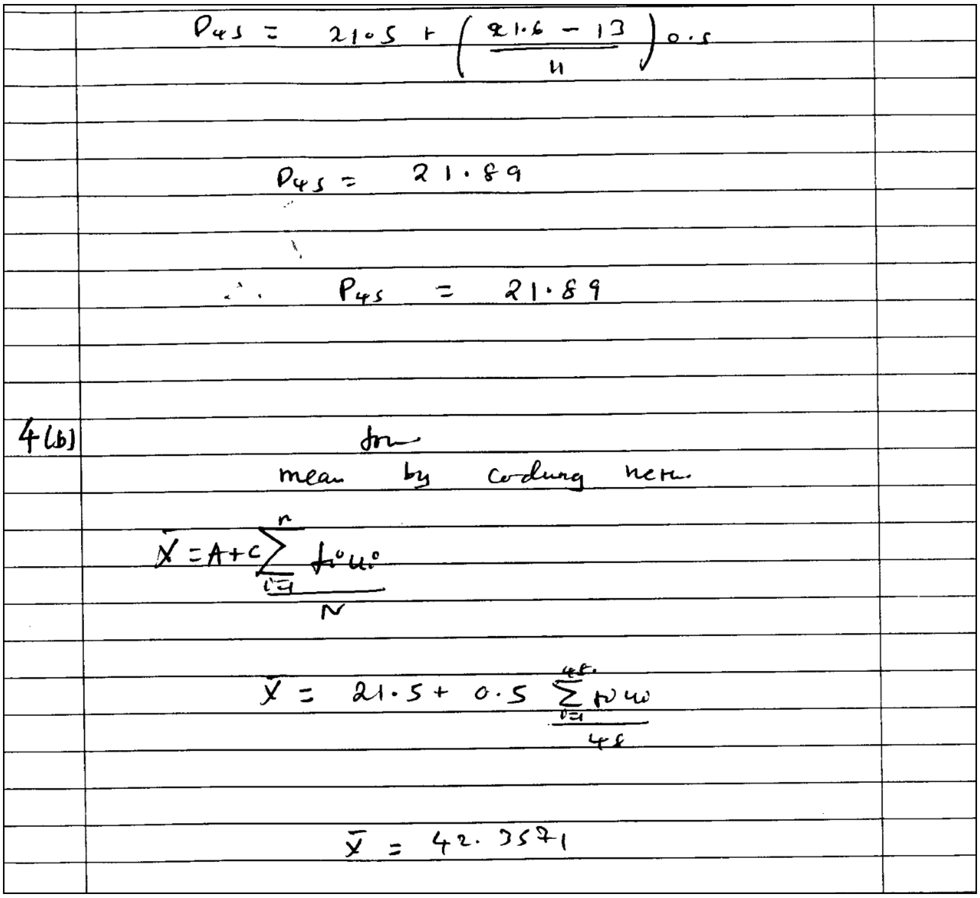
4.	Standard deviation $= c\sqrt{\frac{\sum fu^2}{N} - \left(\frac{\sum fu}{N}\right)^2}$	
	$\delta = 0.5\sqrt{\frac{(109-20)}{78} - \left(\frac{21}{78}\right)^2}$	
	$\delta = 7.21 \times 10^{-1} - 7.626 \times 10^{-2}$	
	$\delta = 6.447 \times 10^{-1}$	
	$\therefore$ Standard deviation is 6.447×10^{-1} Ohms	

Extract 4.1: A sample of correct responses to question 4 of paper 1

In Extract 4.1, the candidate calculated the measures of central tendency and of dispersion from the frequency distribution table using correct formulae.

On the other hand, the analysis of responses indicated that the candidates who performed poorly on this question lacked sufficient knowledge of the measures of central tendency and dispersion. In part (a), although the candidates recalled the correct formula, they failed to compute the lower real limits correctly. Additionally, some candidates were unable to determine the frequencies before and after the modal class. For example, in finding mode, they applied the formula $M = L + \left(\frac{t_1}{t_1 + t_2}\right)c$, then used the value of $t_2 = 2$ instead of 4. Likewise, in finding the median, some candidates selected an incorrect median class. In part (b), although the candidates recalled the correct formula, they failed to construct the frequency distribution table corresponding to the formula. Extract 4.2 presents a sample of an incorrect response from one of the candidates who answered the question incorrectly.

4.	(a). soln.
	Given data from the table
	$\text{mode} = L + \left(\frac{f_1}{f_1+f_2}\right) c$
	i) $\text{mode} = 21.999 + \left(\frac{2}{4}\right) 0.5$
	$\text{mode} = 5.4975$
	$\therefore \text{mode} = 5.5$.
	ii) $\text{median} = L + \left(\frac{N/2 - f_b}{f_w}\right) c$
	$\text{median} = 22.0 + \left(\frac{48/2 - 24}{13}\right) 0.5$
	$\text{median} = 22.$
	$\therefore \text{median} = 22.$
	iii) 45th percentile.
	$P_K = \left(\frac{KN}{100}\right)^{th}$
	$P_K = \left(\frac{45(48)}{100}\right)^{th}$
	$P_{45} = 21.6$
	$L_{QK} + \left(\frac{N/2 - \Sigma f_b}{f_w}\right) c$



$P_{45} = 21.5 + \left(\frac{21.6 - 13}{11}\right)0.5$

$P_{45} = 21.89$

$\therefore P_{45} = 21.89$

4(b) mean by coding method.

$\bar{X} = A + C\frac{\sum_{i=1}^{n} f_i u_i}{N}$

$\bar{X} = 21.5 + 0.5\frac{\sum_{i=1}^{48} f_i u_i}{48}$

$\bar{X} = 42.3571$

Extract 4.2: A sample of incorrect responses to question 4 of paper 1

In Extract 4.2, the candidate correctly recalled the formulae but failed to calculate the mode, median, 45^{th} percentile, mean, and standard deviation.

2.1.5 Question 5: Sets

The question examined the candidates' ability to represent the union and intersection of sets on a number line and to use Venn diagrams in solving real-life problems involving sets. The question was stated as follows:

(a) *Given* $A = \{x \in \square : x \geq 1\}$ *and* $B = \{x \in \square : -5 < x \leq 3\}$, *display each of the following sets on a number line:*

(i) $A \cap B$

(ii) $A \cup B'$

(b) *A survey on the type of cash crops grown in a certain village revealed that out of 230 families, 126 grow cotton, 85 grow sunflower only, 68*

grow cotton and sunflower, 42 grow coffee and sunflower, and 34 grow cotton and coffee only. The number of families who grow cotton only is thrice the number growing coffee only, and *9 grow none of these crops. Using the Venn diagram, determine the number of families growing:*

(i) all three crops.

(ii) exactly one crop.

The data indicates that out of 18,136 (100%) candidates, 3,945 (21.75%) scored from 0 to 3.0 marks, 5,137 (28.33%) candidates scored from 3.5 to 5.5 marks, and 9,054 (49.92%) scored from 6.0 to 10 marks. Figure 6 summarises the overall performance of candidates on this question.

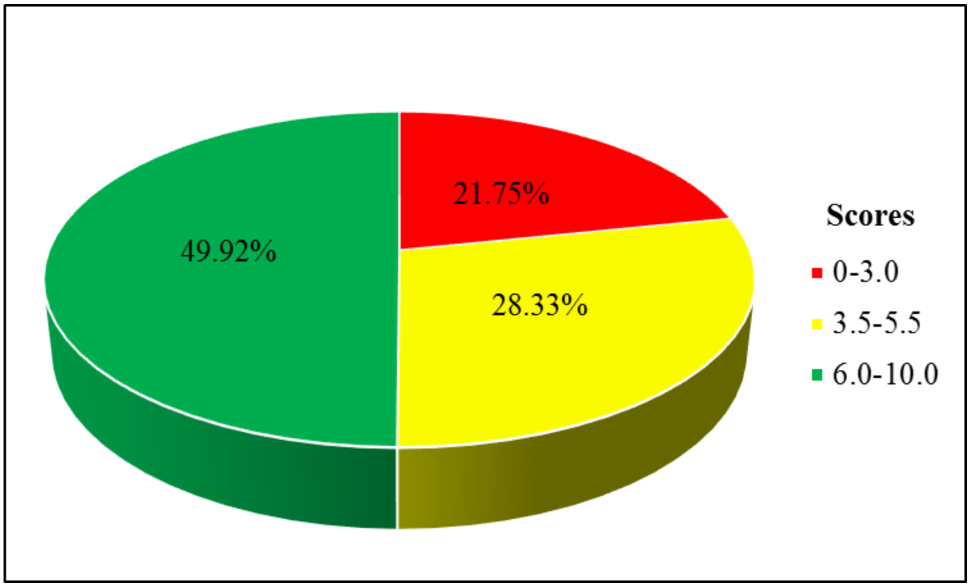
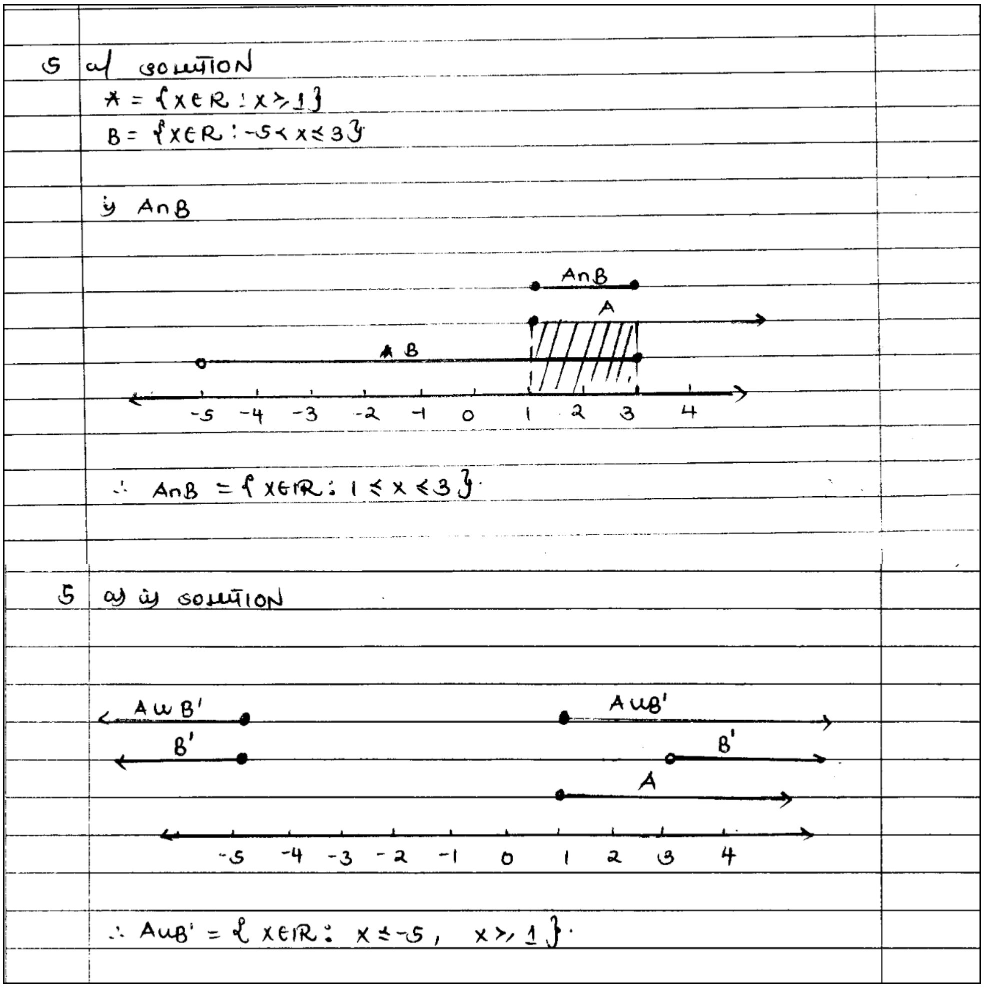


Figure 6: *The Candidates' Performance in Question 5 of Paper 1*

Figure 6 shows that the general performance on this question was good. Further analysis indicates that 4,583 (25.27%) candidates scored more than 8.5 marks. This suggests that the candidates who performed well on this question had good competence in the concepts of sets. In part (a), the candidates represented correctly the sets $A \cap B$ and $A \cup B'$ on a number line using the given sets $A = \{x \in \square : x \geq 1\}$ and $B = \{x \in \square : -5 < x \leq 3\}$ (Extract 5.1). In part (b), the candidates correctly represented the information in a Venn diagram and determined the number of families growing all crops and exactly one crop, which were 40 and 117 families, respectively (Extract 5.1). A sample of correct responses from a candidate who answered this question correctly is shown in Extract 5.1.


5 a) SOLUTION

$A = \{x \in R : x \geq 1\}$

$B = \{x \in R : -5 < x \leq 3\}$

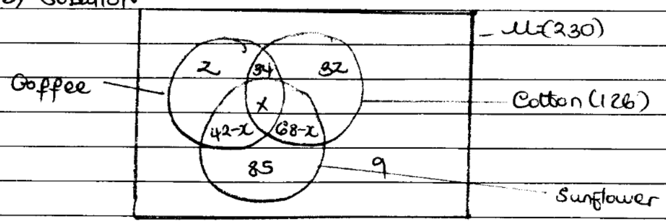
i) $A \cap B$

$\therefore A \cap B = \{x \in R : 1 \leq x \leq 3\}$.

5 a) ii) SOLUTION

$\therefore A \cup B' = \{x \in R : x \leq -5, \; x \geq 1\}$.

5 b) SOLUTION



$34 + 3z + x + 68 - x = 126$

$3z + 102 = 126$

$3z = 24$

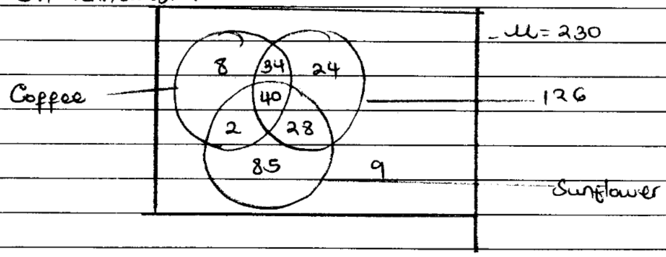
$z = 8$

$8 + 34 + 8 + 24 + x + 68 - x + 42 - x + 85 + 9 = 230$

$-x + 270 = 230$

$x = 40$

On venn diagram



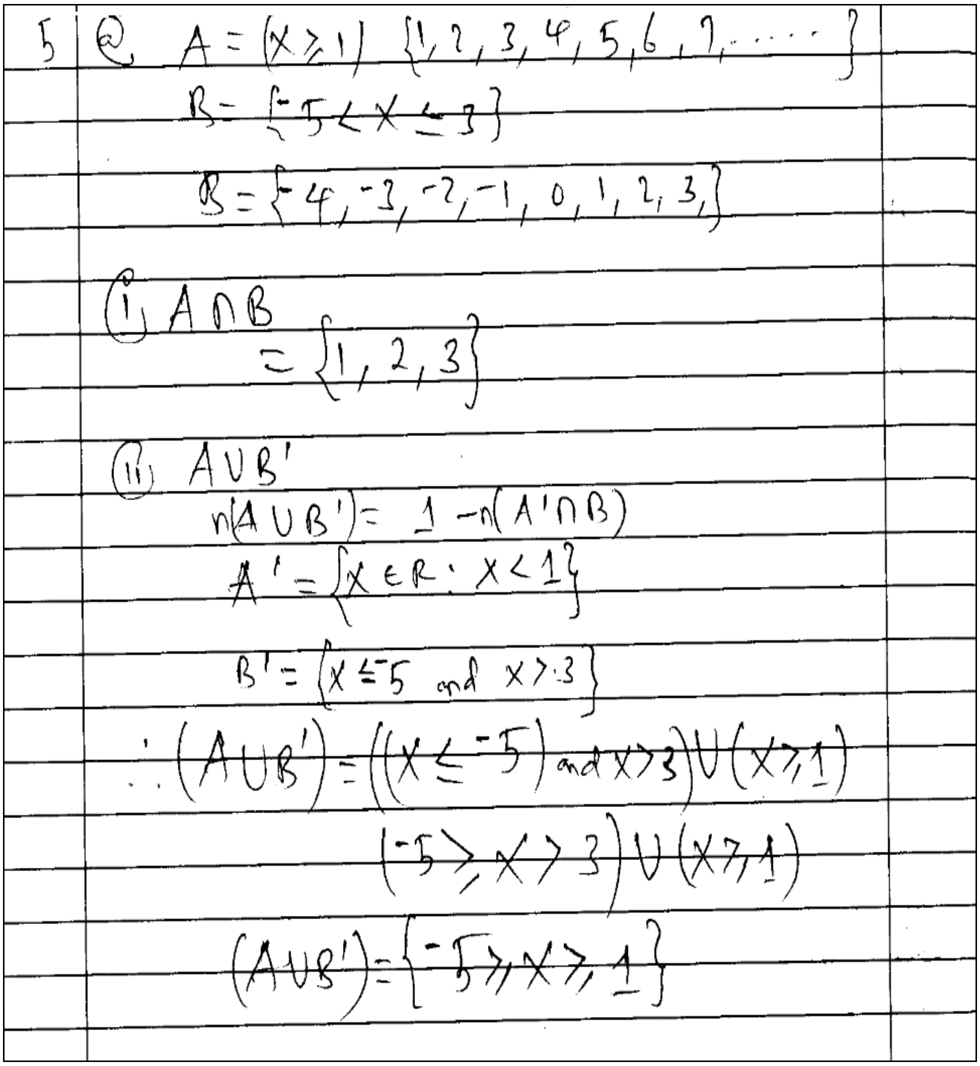
i) 40 families

ii) 8 + 24 + 85 =
117 families

Extract 5.1: A sample of correct responses to question 5 of paper 1

In Extract 5.1, part (a), the candidate displayed the given sets on the number line and correctly indicated $A \cap B$ and $A \cup B'$. In part (b), the candidate represented the given information in a Venn diagram and used it to determine the number of families growing all crops and those growing exactly one crop.

Although overall performance was good, some candidates failed to obtain correct answers due to insufficient knowledge of how to represent sets on a number line and how to use Venn diagrams to solve set-related problems. In part (a), the candidates misinterpreted the question and therefore responded contrary to its requirements. For example, they treated the given sets as subsets of the set of integers instead of the set of real numbers. Then, they wrote $A = \{1, 2, 3, 4, 6, 7, ...\}$ and $B = \{-4, -3, -2, -1, 0, 1, 2, 3\}$ which led to getting wrong answers such as $A \cap B = \{1,2,3\}$. In part (b), the candidates incorrectly represented the information in a Venn diagram, which resulted in wrong answers. Extract 5.2 is a sample of incorrect responses from one of the candidates who answered the question incorrectly.



5 (a) $A = (x \geq 1)$ $\{1, 2, 3, 4, 5, 6, 7, ...\}$

$B = \{-5 < x \leq 3\}$

$B = \{-4, -3, -2, -1, 0, 1, 2, 3,\}$

(i) $A \cap B$

$= \{1, 2, 3\}$

(ii) $A \cup B'$

$n(A \cup B') = 1 - n(A' \cap B)$

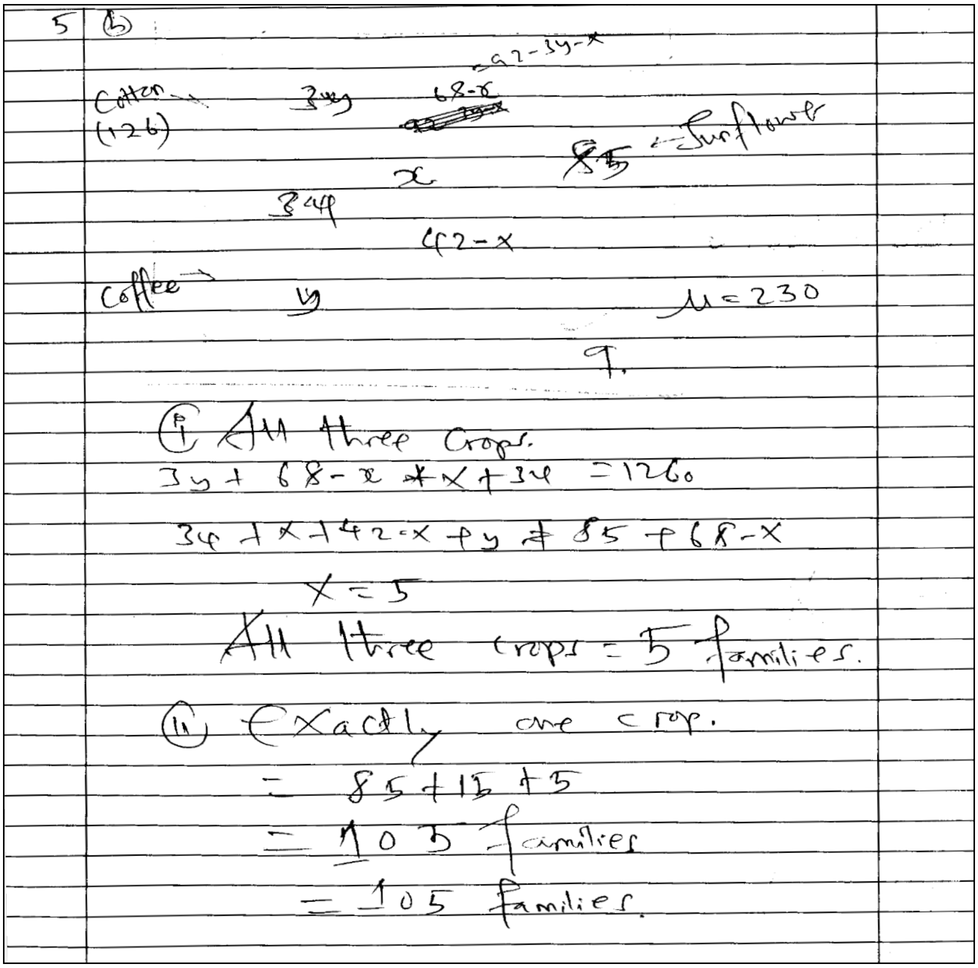
$A' = \{x \in R : x < 1\}$

$B' = \{x \leq -5 \text{ and } x > 3\}$

$\therefore (A \cup B') = ((x \leq -5) \text{ and } x > 3) \cup (x \geq 1)$

$(-5 \geq x > 3) \cup (x \geq 1)$

$(A \cup B') = \{-5 \geq x \geq 1\}$



5 (b)

Cotton (126) 3y 68-x 92-3y-x

85 Sunflower

x

34

42-x

Coffee y U=230

(i) All three crops.

3y + 68 - x + x + 34 = 126

34 + x + 42 - x + y + 85 + 68 - x

x = 5

All three crops = 5 families.

(ii) Exactly one crop.

= 85 + 15 + 5

= 105 families

= 105 families.

Extract 5.2: A sample of incorrect responses to question 5 of paper 1

In Extract 5.2, part (a), the candidate listed the integers instead of displaying the inequalities on a number line. In part (b), the candidate failed to represent the information in a Venn diagram; thus, he/she could not obtain the correct answers.

2.1.6 Question 6: Functions

The question examined the candidates' competence in drawing graphs of cubic and rational functions and how to determine their domains and ranges. The question was as follows:

(a) *Given the function* $f(x) = x^3 + 3x^2 - 2x - 6$, *draw the graph of* $f(x)$ *and hence, state its domain and range.*

(b) *Given that,* $g(x) = \dfrac{x+1}{2x^2 + 5x - 3}$,

(i) *find the vertical asymptotes.*

(ii) *Sketch the graph of* $g(x)$ *and determine the domain and range of* $g(x)$.

The data analysis depicts that 18,136 (100%) candidates attempted this question. Among them, 981 (5.41%) scored 0 to 3.0 marks, 3,069 (16.92%) scored 3.5 to 5.5 marks, and 14,086 (77.67%) candidates scored 6.0 to 10 marks. Thus, the performance of candidates on this question was generally good, as 94.59% passed. Figure 7 summarises the candidates' performance on this question.

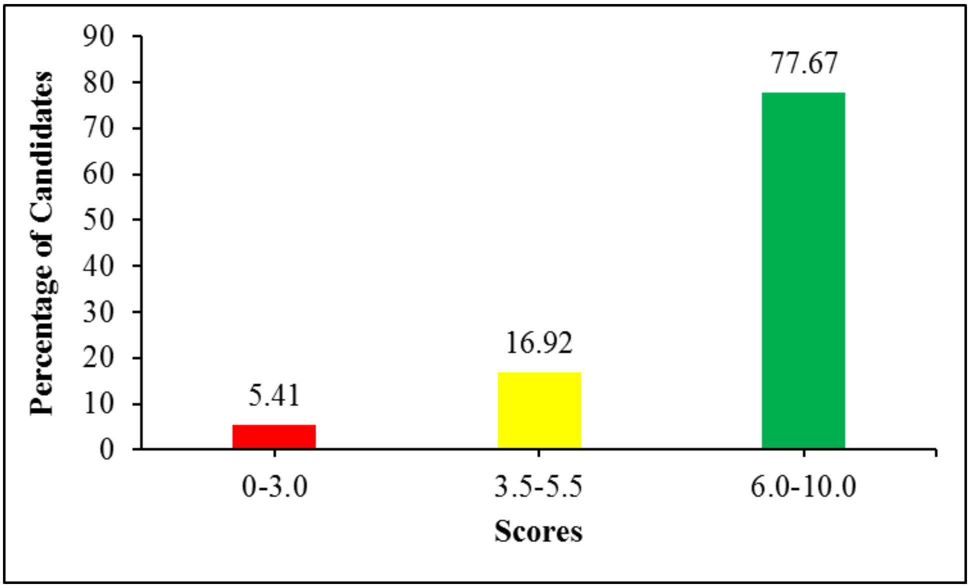
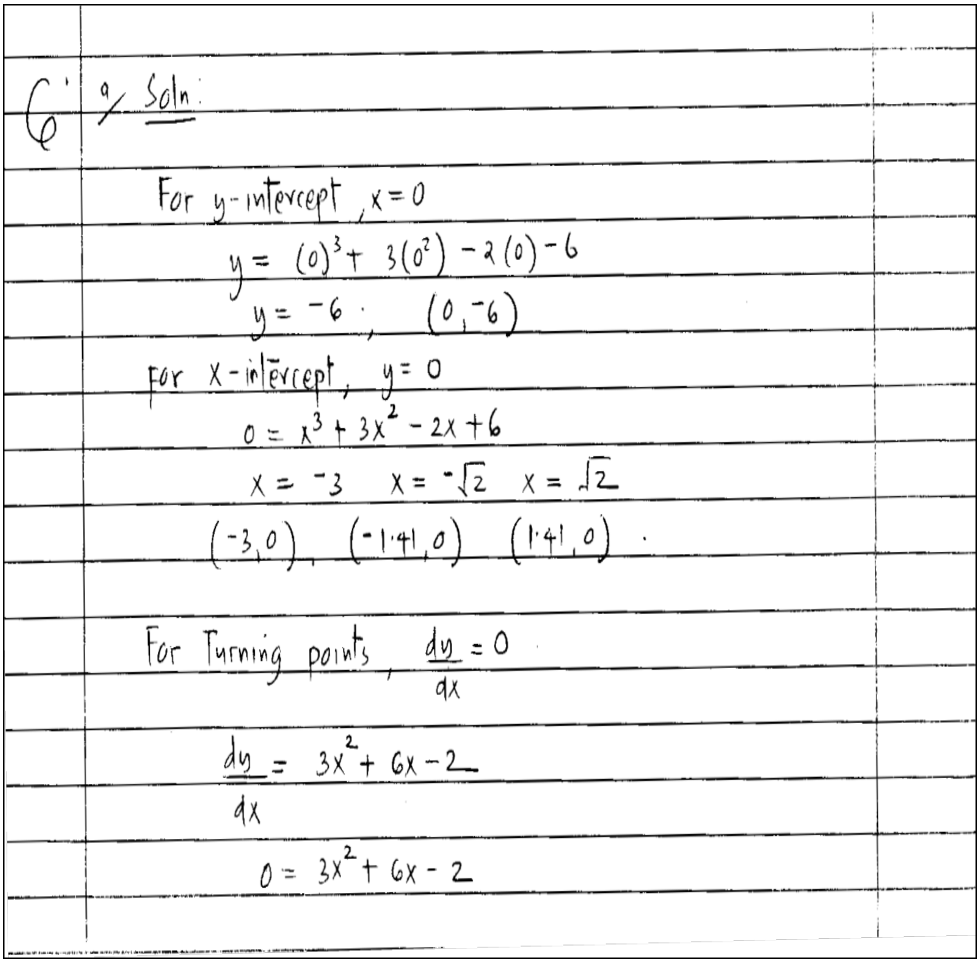


Figure 7: *The Candidates' Performance on Question 6 in Paper 1*

The data also show that 381 (2.10%) candidates scored full marks due to their adequate knowledge of the concepts of cubic and rational functions. In part (a), the candidates determined the $x-$intercepts as $(-3,0)$, $\left(\frac{-3}{2},0\right)$, and $\left(\frac{3}{2},0\right)$, then $y-$intercept $(0,-6)$. Thereafter, they determined the stationary points of the curve by differentiating the given function, equated the derivative to zero, and finally obtained the minimum and maximum turning points as $(0.3,-6.3)$ and $(-2.3, 2.3)$, respectively. Then, they correctly drew the graph using this information and stated its domain and range correctly (Extract 6.1). In part (b)(i), the candidates found the vertical asymptotes by equating the denominator to zero and obtained $x = -3$ and $x = \frac{1}{2}$. In part (b) (ii), the candidates drew the graph using the vertical asymptotes obtained in (b) (i) and other properties such as the horizontal asymptote, which is $y = 0$, and x- and y- intercepts equal to $(-1, 0)$ and $\left(0, \frac{-1}{3}\right)$, respectively. Extract 6.1 shows a sample of correct responses from a candidate who answered this question correctly.



6. a/ Soln:

For y-intercept, $x = 0$

$y = (0)^3 + 3(0^2) - 2(0) - 6$

$y = -6$; $(0, -6)$

For x-intercept, $y = 0$

$0 = x^3 + 3x^2 - 2x + 6$

$x = -3$ $x = -\sqrt{2}$ $x = \sqrt{2}$

$(-3, 0)$, $(-1.41, 0)$ $(1.41, 0)$.

For Turning points, $\frac{dy}{dx} = 0$

$\frac{dy}{dx} = 3x^2 + 6x - 2$

$0 = 3x^2 + 6x - 2$

6 a) $x = 0.29$ $x = -2.29$ Points $(0.29, -6.3)$

$y = -6.3$ $y = 2.3$ $(-2.29, 2.3)$

$\therefore$ Domain = $\{x \in \mathbb{R}\}$.

Range = $\{y \in \mathbb{R}\}$

b) Soln:

$$g(x) = \frac{x+1}{2x^2+5x-3}$$

$$g(x) = \frac{x+1}{(2x-1)(x+3)}.$$

i) Vertical asymptote,

$$(2x^2+5x-3) = 0$$

$$(2x-1)(x+3) = 0$$

$$2x-1 = 0 \qquad x+3 = 0$$

$$x = 1/2 \qquad x = -3.$$

$\therefore$ Vertical asymptotes are $x = 1/2$, $x = -3$.

ii) Soln:

For horizontal asymptote,

$$y = \frac{x/x^2 + 1/x^2}{2x^2/x^2 + 5x/x^2 - 3/x^2} \quad \text{as } x \to \infty$$

$$y = \frac{0}{2} = 0.$$

$$y = 0.$$

6. b) ii) For y-intercept, $x = 0$

$$y = \frac{0+1}{2(0^2)+5(0)-3} = \frac{-1}{3} \quad (0, -1/3).$$

For x-intercept, $y = 0$

$$0 = x+1$$

$$x = -1 \quad (-1, 0).$$

for turning point, $\frac{dy}{dx} = 0$,

$$\frac{dy}{dx} = \frac{(2x^2+5x-3) - (x+1)(4x+5)}{(2x^2+5x-3)^2}$$

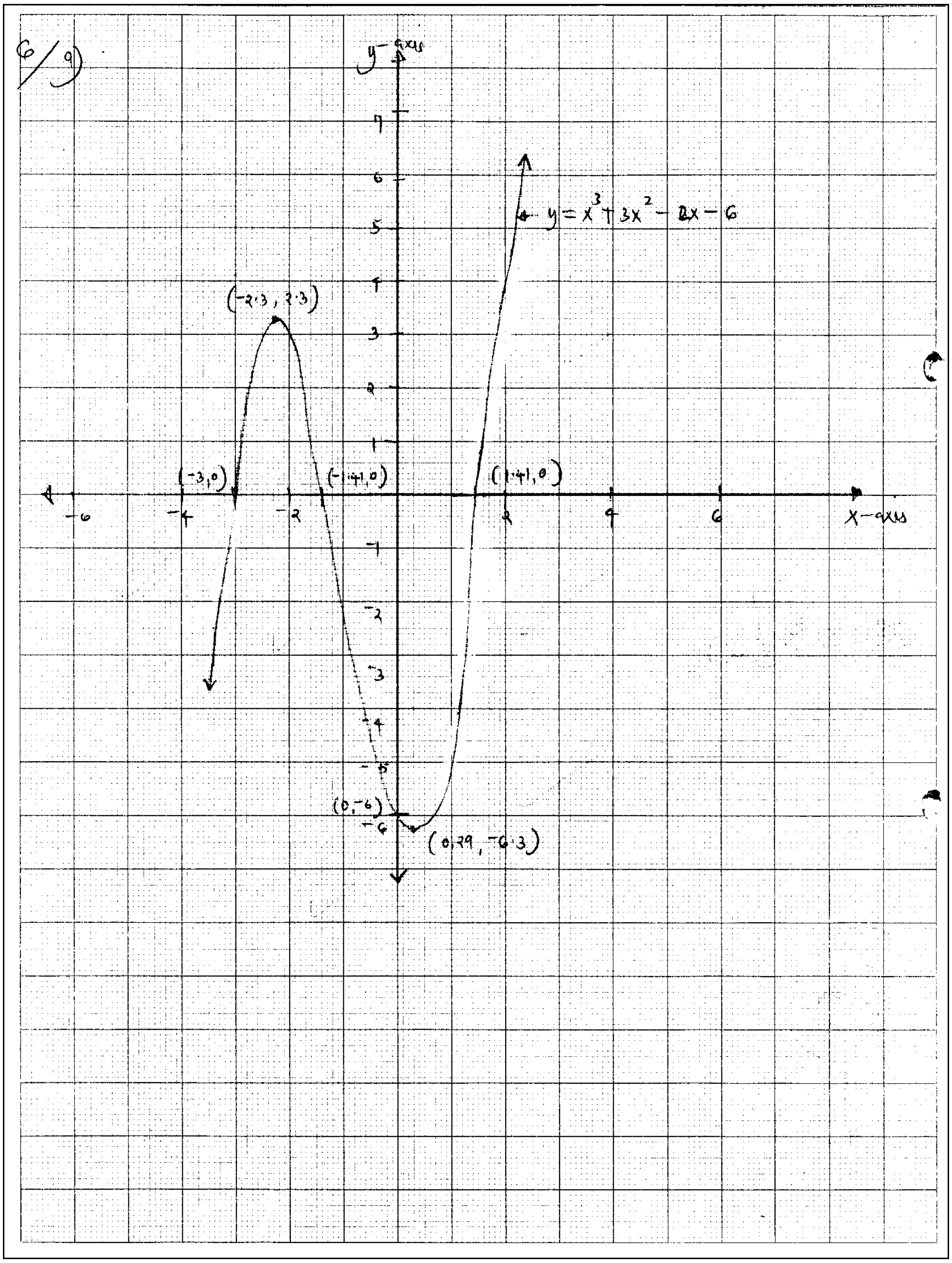
$$0 = 2x^2+5x-3 - 4x^2-9x-5$$

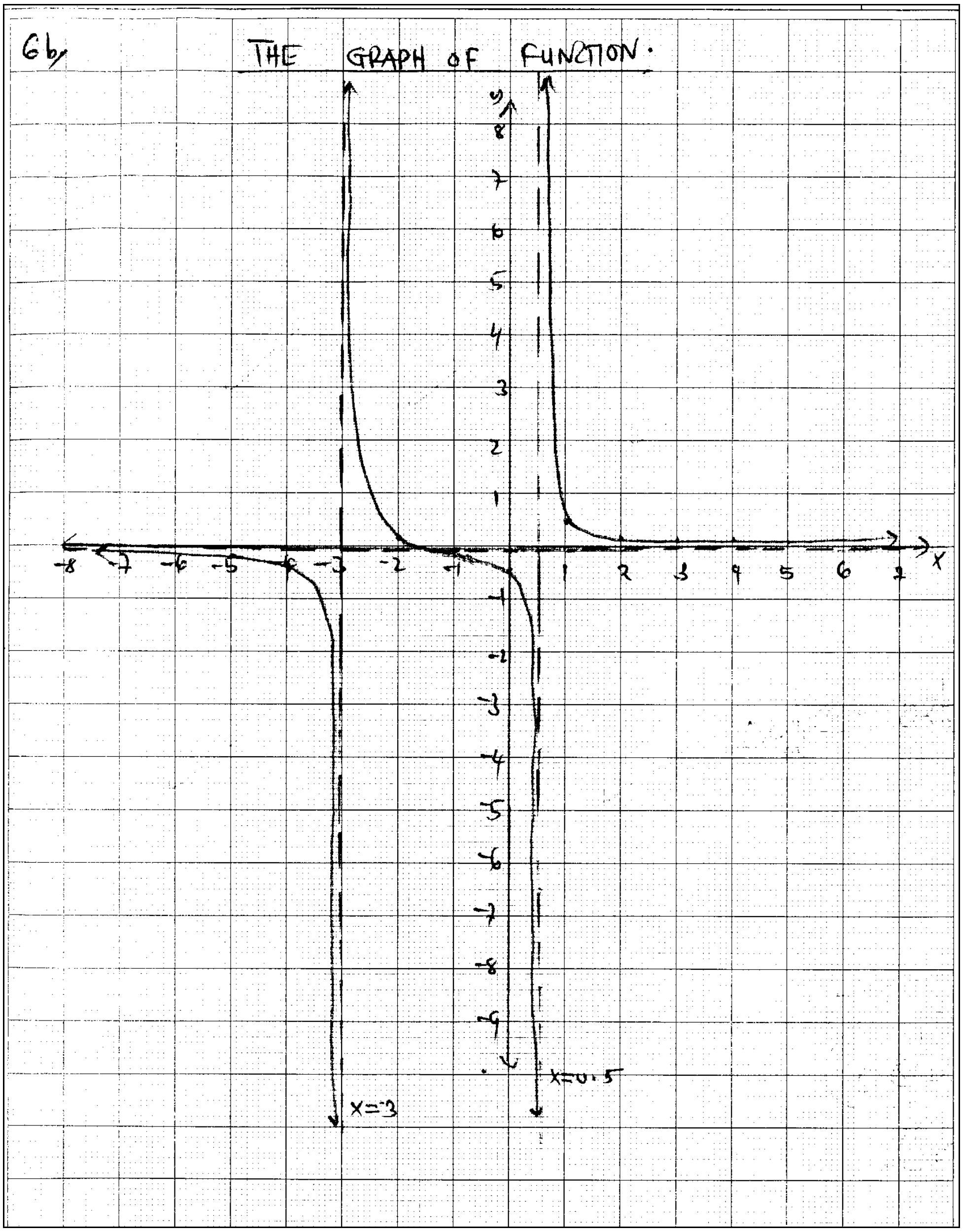
$$0 = -2x^2-4x-8$$

$$0 = 2x^2+4x+8$$

$\therefore$ Domain $= \{x \in \mathbb{R},\ x \neq 1/2,\ x \neq -3\}$.

Range $= \{y \in \mathbb{R}\}$


6/9)
y-axis
x-axis
y = x³ + 3x² − 2x − 6
(−2·3, 3·3)
(−3,0)
(−1·41,0)
(1·41,0)
(0,−6)
(0·29, −6·3)



Extract 6.1: A sample of correct responses to question 6 of paper 1

In Extract 6.1, the candidate drew the correct graphs using intercepts, stationary points, and asymptotes. From the graphs, he/she stated their domains and ranges.

On the other hand, some candidates faced challenges when responding to this question due to a lack of competence in the concepts of cubic and rational functions. In part (a), some candidates used a table of values, which did not cover all the necessary information, such as turning points and asymptotes. The prepared tables of values led them to sketch incorrect graphs. In part (b), the candidates failed to determine the vertical and horizontal asymptotes. They tried to use a table of values, which did not provide the necessary information to sketch the graph correctly. Extract 5.2 is a sample of incorrect responses from one of the candidates who answered the question incorrectly.

6. a) Given

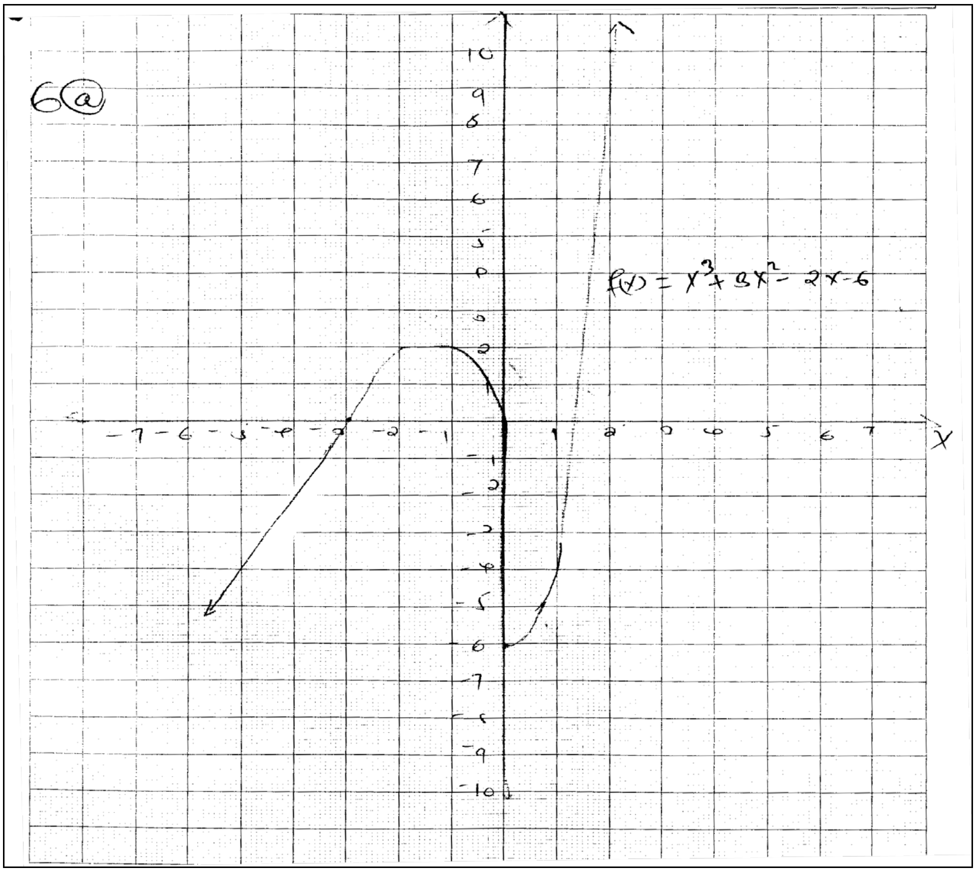
$f(x) = x^3 + 3x^2 - 2x - 6$

Table values

x	-4	-3	-2	-1	0	1	2	3	4
y	-14	0	2	-2	0	-4	10	42	98

Domain = {All real numbers}

Range = {All real numbers}



Extract 6.2: A sample of incorrect responses to question 6 of paper 1

In Extract 6.2, the candidate incorrectly used a table of values, failed to determine the vertical asymptotes and stationary points, and consequently drew an incorrect graph.

2.1.7 Question 7: Numerical Methods

The question examined the candidates' ability to derive the secant formula and apply both the secant and Newton-Raphson formulas to approximate the roots of functions. The question was stated as follows:

(a) *Derive the secant formula for approximating the roots of the equation* $f(x) = 0$.

(b) *Using the formula derived in part (a) and the interval (1.5, 2), perform two iterations to solve the equation* $x^3 - 3 = 0$ *correct to two decimal places.*

(c) *By using the Newton-Raphson method and* $x_0 = 5$, *perform two iterations to approximate the solution of the equation* $x^3 - 4x^2 - x - 12 = 0$ *correct to three significant figures.*

This question was attempted by 18,136 (100%) candidates. The data analysis shows that 2,810 (15.49%) scored 0 to 3.0 marks, 3,617 (19.95%) scored 3.5 to 5.5 marks, and 11,709 (64.56%) scored 6.0 to 10 marks. This indicates that the general performance of candidates on this question was good. Figure 8 summarises the overall performance of the candidates with this question.

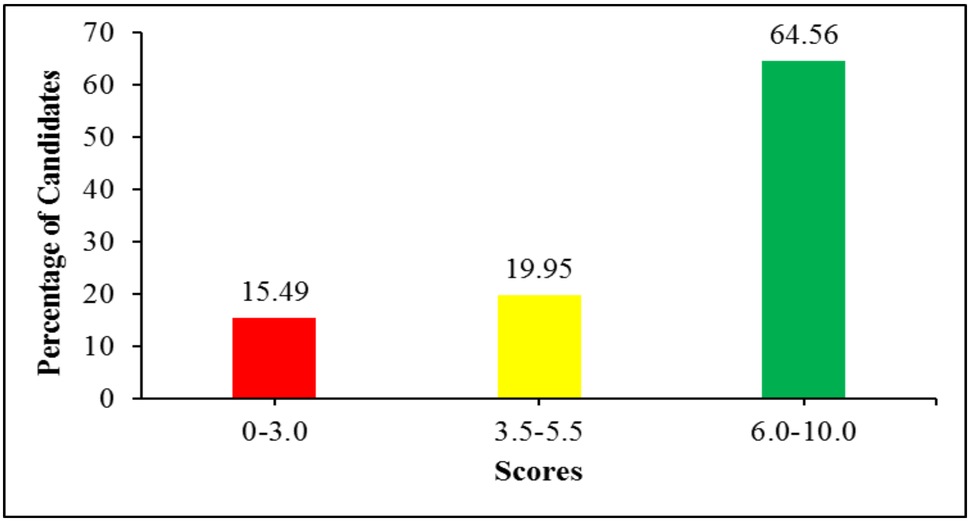


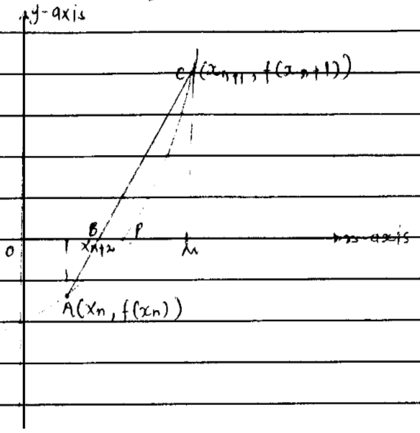
Figure 8: *The Candidates' Performance in Question 7 of Paper 1*

Further analysis of the data shows that 3,975 (21.92%) candidates scored 10 marks. This was attributed to their competence in solving problems using numerical methods. In part (a), they correctly derived the secant formula with the help of a sketch. They sketched a simple graph of the curve $f(x)$ that crosses the $x-$axis and a secant line that crosses the $x-$axis at $(x_{n+2}, 0)$ and intersects the curve at two points, $(x_n, f(x_n))$ and $(x_{n+1}, f(x_{n+1}))$ (Extract 7.1). Using the concept of the slope of a straight line, they derived the secant formula correctly to get $x_{n+2} = x_n + \frac{f(x_n)(x_n - x_{n+1})}{f(x_{n+1}) - f(x_n)} \equiv x_{n+2} = x_{n+1} + \frac{f(x_{n+1})(x_n - x_{n+1})}{f(x_{n+1}) - f(x_n)}$.

In part (b), the candidates applied the formula derived in (a) to solve the equation $x^3 - 3 = 0$ by performing two iterations. They obtained $x_2 = 1.459$ in the first iteration and $x_3 = 1.4469$ in the second iteration. Then, they rounded off the result to two decimal places to get $x_3 = 1.45$. In part (c), the candidates were able to apply the Newton-Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ to approximate the solution of the equation $x^3 - 4x^2 - x - 12 = 0$. They performed two iterations and got $x_1 = 4.7647$ in the first iteration and $x_2 = 4.74$ in the second iteration, correct to two decimal places as required. Extract 7.1 is a sample of correct responses from one of the candidates who had adequate skills in attempting this question.

7a) Solutions:

From the sketch graph;



A, B and C are collinear points.

slope $\bar{BC}$ = slope $\bar{AC}$

$B(x_{n+2}, 0)$

$$\frac{f(x_{n+1}) - 0}{x_{n+1} - x_{n+2}} = \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n}$$

OR; slope $\bar{AB}$ = slope $\bar{AC}$

$$\frac{f(x_n) - 0}{x_n - x_{n+2}} = \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n}$$

$$\frac{f(x_n)(x_{n+1} - x_n)}{f(x_{n+1}) - f(x_n)} = x_n - x_{n+2}$$

$$x_{n+2} = x_n - \left[\frac{x_{n+1} - x_n}{f(x_{n+1}) - f(x_n)}\right] f(x_n)$$

∴ The secant for approximating roots of an equation is;

$$x_{n+2} = x_n - \left[\frac{x_{n+1} - x_n}{f(x_{n+1}) - f(x_n)}\right] f(x_n)$$

7b) Solutions

$$x^3 - 3 = 0$$

$$f(x) = x^3 - 3$$

$$X_{n+2} = X_n - \left[\frac{X_{n+1} - X_n}{f(X_{n+1}) - f(X_n)}\right] f(X_n)$$

$X_0 = 1.5$

$X_1 = 2$

$f(x) = x^3 - 3$

$f(X_0) = 0.375$

$f(X_1) = 5$

1st iteration;

$X_0 = 1.5 \qquad X_1 = 2$

$f(X_0) = 0.375 \qquad f(X_1) = 5$

$$X_2 = 1.5 - \left[\frac{2 - 1.5}{5 - 0.375}\right] 0.375$$

$X_2 = 1.46.$

2nd iteration;

$X_1 = 2 \qquad X_2 = 1.46$

$f(X_1) = 5 \qquad f(X_2) = 0.11$

$$X_3 = 2 - \left[\frac{1.46 - 2}{0.11 - 5}\right] 5$$

$X_3 = 1.45$

∴ The ~~value~~ of root of the equation is 1.45

c) Solutions

$$x^3 - 4x^2 - x - 12 = 0$$

$$f(x) = x^3 - 4x^2 - x - 12$$

$$f(x_n) = x_n^3 - 4x_n^2 - x_n - 12$$

$$f'(x_n) = 3x_n^2 - 8x_n - 1$$

From;

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

7c)	$x_{n+1} = x_n - \dfrac{x_n^3 - 4x_n^2 - x_n - 12}{3x_n^2 - 8x_n - 1}$	
	$x_{n+1} = \dfrac{3x_n^3 - 8x_n^2 - x_n - x_n^3 + 4x_n^2 + x_n + 12}{3x_n^2 - 8x_n - 1}$	
	$x_{n+1} = \dfrac{2x_n^3 - 4x_n^2 + 12}{3x_n^2 - 8x_n - 1}$ ✓	
	$x_0 = 5$	
	1^{st} iteration	
	$x_1 = \dfrac{2x_0^3 - 4x_0^2 + 12}{3x_0^2 - 8x_0 - 1}$	
	$x_1 = 4.76$	
	2^{nd} iteration	
	$x_2 = \dfrac{2x_1^3 - 4x_1^2 + 12}{3x_1^2 - 8x_1 - 1}$	
	$x_2 = 4.744$	
	$\therefore$ athe approximate solution is $x = 4.744$	

Extract 7.1: A sample of correct responses to question 7 of paper 1

In Extract 7.1, the candidate applied appropriate procedures in deriving the secant formula and correctly used both the secant and Newton-Raphson formulae to approximate the roots of the given functions.

In spite of the good performance shown by most candidates, some faced challenges when responding to this question. In part (a), some candidates used an incorrect sketch to derive the secant formula and therefore ended up with an incorrect result. Others applied a wrong concept by attempting to derive the secant formula from Taylor's theorem. In part (b), there were candidates who failed to develop the function $f(x) = x^3 - 3$ from the given equation $x^3 - 3 = 0$. Thus, they wrote $x^3 = 3$ and ended up getting wrong answers. In part (c), the candidates failed to recall the correct Newton-Raphson formula. They used the incorrect formula $x_{n+1} = x_n - \dfrac{f'(x_n)}{f(x_n)}$ and obtained the wrong iterative values $x_1 = 0.75$ and $x_2 = 0.3856$. Others applied the secant formula instead of the Newton-Raphson formula to find the roots of the given equation. Extract 7.2 is a sample of incorrect responses from a candidate who answered the question incorrectly.

07 (a) from tylor's series neglecting the highest value

~~X-h=~~

$$f(x-h) = f(x-h) + hf'\frac{(x-h)}{1!} + h^2 f''\frac{(x-h)}{2!}$$

but $h = 0$ h

$$f(x) = x_{n+1} - \frac{f(x_n)(x_{n+1} - x_n)}{f(x_{n+1}) - f(x_n)}$$

$$x_{n+2} = x_{n+1} - \frac{f(x_n)(x_{n+1} - x_n)}{f(x_{n+1}) - f(x_n)}$$

Proved.

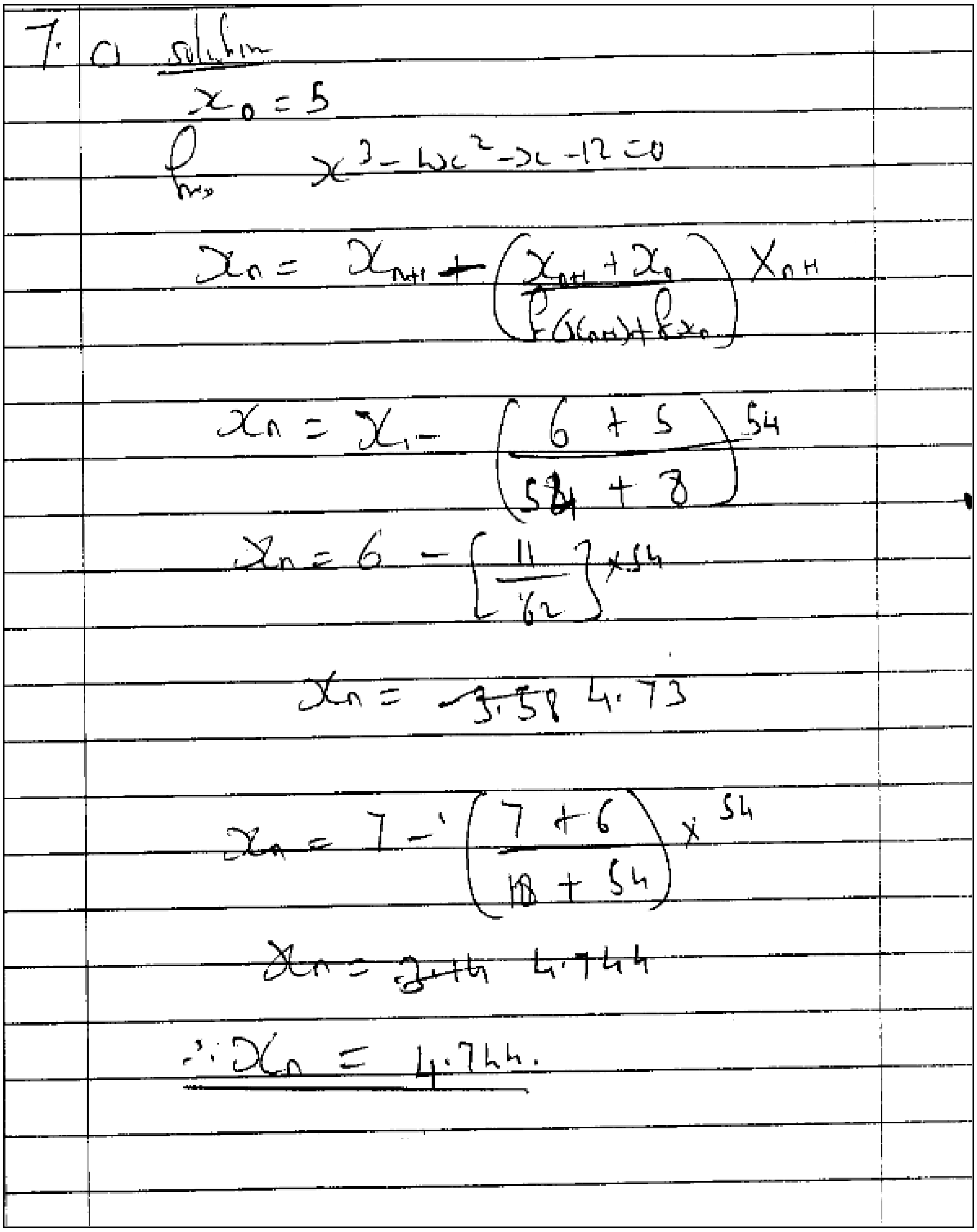
07 (b) Given

$x_0 = 1.5$, $x_1 = 2$.

and

$x^3 - 3 = 0$ but $f(0) = 0$

$f(x) = x^3 - 3$



7. a solution

$x_0 = 5$

$f(x)$ $x^3 - 4x^2 - x - 12 = 0$

$x_n = x_{n+1} + \left(\frac{x_{n+1} + x_0}{f(x_{n+1}) + f(x_0)}\right) x_{n+1}$

$x_n = x_1 - \left(\frac{6 + 5}{54 + 8}\right) 54$

$x_n = 6 - \left[\frac{11}{62}\right] \times 54$

$x_n = \sout{3.58}\ 4.73$

$x_n = 7 - \left(\frac{7 + 6}{18 + 54}\right) \times 54$

$x_n = \sout{3.44}\ 4.744$

$\therefore x_n = 4.744.$

Extract 7.2: A sample of incorrect responses to question 7 of paper 1

In Extract 7.2, part (a), the candidate applied the wrong concept by attempting to derive the secant formula from Taylor's theorem. In part (c), the candidate applied the secant formula instead of the Newton-Raphson formula to approximate the solution of the equation $x^3 - 4x^2 - x - 12 = 0$ as instructed.

2.1.8 Question 8: Coordinate Geometry I

The question examined the candidates' ability to calculate the perpendicular distance of a point from a line, find equations for the angle bisectors between two lines, and calculate the length of a tangent to a circle from a fixed point. The question was as follows:

(a) Calculate the perpendicular distance of a point $(3, -5)$ *from the line* $2x - y = 1$.

(b) Obtain the equations to the bisectors of angles between the lines $3x + 4y = 12$ *and* $4x - 3y = 6$.

(c) Find the length of a tangent to the circle $x^2 + y^2 + 2x + 2y - 7 = 0$ *from the point* $(2, 3)$.

This question was attempted by 18,136 (100%) candidates, of whom 2,004 (11.05%) scored 0 to 3.0 marks and 16,132 (88.95%) scored 3.5 to 10 marks. Therefore, the performance of candidates on this question was generally good. Figure 9 summarises the candidates' performance on this question.

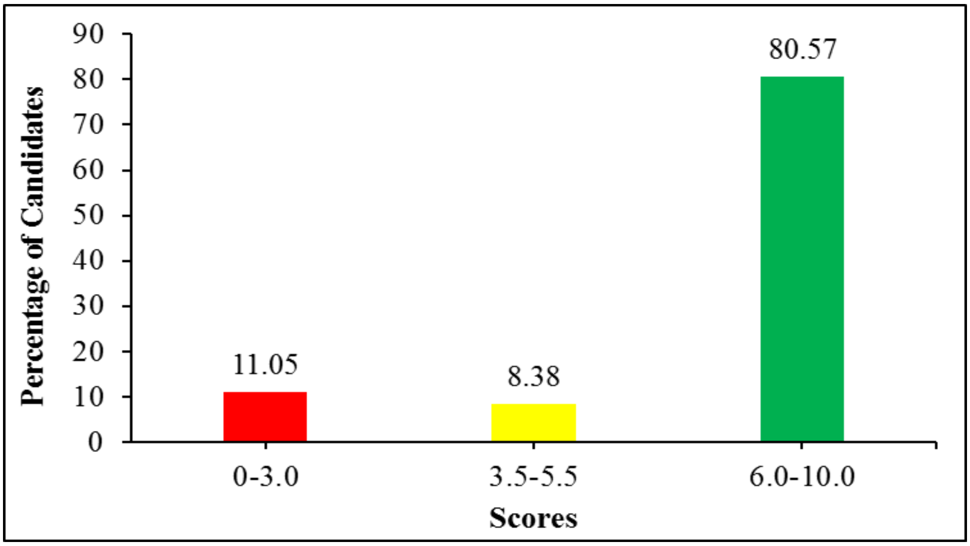


Figure 9: *The Candidates' Performance in Question 8 of Paper 1*

In this question, 7,986 (44.03%) candidates responded to all parts correctly. In part (a), the candidates recalled the correct formula for calculating the perpendicular distance of a point (x_1, y_1) from a line, $ax + by + c = 0$, that is, $d = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$. They used this formula to calculate the distance of the point $(3,-5)$ from the line $2x - y = 1$ and obtained the correct answer, $d = 2\sqrt{5}$ units. In part (b), the candidates correctly recalled and applied the formula for finding the equations of the angle bisectors between two lines. They wrote $\left| \frac{a_1x_1 + b_1y_1 + c_1}{\sqrt{(a_1)^2 + (b_1)^2}} \right| = \left| \frac{a_2x_2 + b_2y_2 + c_2}{\sqrt{(a_2)^2 + (b_2)^2}} \right|$ and substituted the values extracted from the lines $3x + 4y = 12$ and $4x - 3y = 6$ to obtain $\left| \frac{3x + 4y - 12}{\sqrt{(3)^2 + (4)^2}} \right| = \left| \frac{4x - 3y - 6}{\sqrt{(4)^2 + (-3)^2}} \right|$. Finally, they simplified the expressions to obtain the required bisectors $x - 7y + 6 = 0$ and $7x + y - 18 = 0$. In part (c), the candidates recalled the correct formula $L = \sqrt{x_1{}^2 + y_1{}^2 + ax_1 + by_1 + c_1}$ for finding the length of a tangent from a point $A(x_1, y_1)$ to a circle $x^2 + y^2 + ax + by + c = 0$. Upon substituting the coordinates of the point $(2, 3)$ into the formula, they obtained the correct length, $L = 4$ units. Extract 8.1 shows a sample of correct responses from one of the candidates who had adequate skills in attempting this question.

8a)	Solution.	
	$(3,-5)$	
	line; $2x - y = 1$	
	distance $= \left\| \frac{ax + by + c}{\sqrt{a^2 + b^2}} \right\|$	
	$= \left\| \frac{2x - y - 1}{\sqrt{4+1}} \right\|$	
	$= \left\| \frac{2(3) + 5 - 1}{\sqrt{5}} \right\|$	
	$= \left\| \frac{6 + 5 - 1}{\sqrt{5}} \right\|$	
	$= \frac{10}{\sqrt{5}}$	
	$= 2\sqrt{5}$ ∴ The perpendicular distance is $2\sqrt{5}$ units	

8b) solutions

line 1; $3x + 4y = 12$

line 2; $4x - 3y = 6$

from;

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

$$\frac{3x + 4y - 12}{\sqrt{9 + 16}} = \pm \frac{4x - 3y - 6}{\sqrt{16 + 9}}$$

$$\frac{3x + 4y - 12}{5} = \pm \frac{4x - 3y - 6}{5}$$

for +ve;

$$\frac{3x + 4y - 12}{5} = \frac{4x - 3y - 6}{5}$$

$$3x + 4y - 12 = 4x - 3y - 6$$

$$x - 7y + 6 = 0$$

for -ve

$$\frac{3x + 4y - 12}{5} = \frac{-4x + 3y + 6}{5}$$

$$3x + 4y - 12 = -4x + 3y + 6$$

$$7x + y - 18 = 0$$

∴ The equations of the bisectors are;

$x - 7y + 6 = 0$ and

$7x + y - 18 = 0$

c) Solutions

length of tangent from circle

$x^2 + y^2 + 2x + 2y - 7 = 0$ from point $(2, 3)$

$$\text{length} = \sqrt{x^2 + y^2 + 2gx + 2fy + c}$$

$$= \sqrt{x^2 + y^2 + 2x + 2y - 7}$$

at $(2, 3)$

$$= \sqrt{2^2 + 3^2 + 2(2) + 2(3) - 7}$$

8c) $\text{length} = \sqrt{4 + 9 + 4 + 6 - 7}$

$$= \sqrt{16}$$

$$= 4$$

∴ The length of the tangent is 4 units.

Extract 8.1: A sample of correct responses to question 8 of paper 1

In Extract 8.1, part (a), the candidate applied the formula for calculating the perpendicular distance of a point from a line and correctly calculated the required distances. In part (b), he/she applied the appropriate concept to find the angle bisectors between the given lines. In part (c), the candidate applied the correct formula for calculating the length of a tangent (L) to the circle from the point of tangency, $T(x, y)$, to the point $A(x_1, y_1)$.

On the other hand, 798 (4.40%) candidates scored zero on this question, as they failed to apply appropriate concepts and correct formulae. In part (a), the candidates applied incorrect formulae, such as $d = \left|\frac{ax+by+c}{\sqrt{x^2+y^2}}\right|$, which resulted in $d = \left|\frac{2x-y-1}{\sqrt{x^2+y^2}}\right|$, and produced a wrong answer, $d = \frac{5}{17}\sqrt{34}$. In part (b), some candidates misinterpreted the question; instead of finding the equations of the angle bisectors between the given lines, they calculated the angle between them (Extract 8.2). In part (c), the candidates used an incorrect concept when trying to find the length of the tangent. They attempted to find the equation of the tangent and use it to calculate the length. They wrote $d = \left|\frac{mx-y+c}{\sqrt{m^2+1}}\right|$, then expressed c in terms of m, and finally ended up with $d = \left|\frac{2m-3+1}{\sqrt{m^2+1}}\right|$. Others determined the equation of a tangent by first finding the gradient m_1 using the centre of the circle $(-1,-1)$ and a point $(2,3)$ to obtain $m_1 = \frac{4}{3}$. Thereafter, for perpendicular lines, $m_1m_2 = -1$, they determined $m_2 = -\frac{3}{4}$, which they used to find the equation of the tangent through the point $(2,3)$ and obtained $4y+3x-9=0$. These responses indicate that the candidates had not acquired the intended competence during the learning process. Extract 8.2 is a sample of incorrect responses from one of the candidates who failed to answer the question correctly.

8a	Soln,	
	point = (3, -5)	
	equation	
	$2x - y = 1$	
	$2x - y - 1 = 0$	
	from.	
	$f(x, y) = \frac{ax + by + c}{\sqrt{a^2 + b^2}}$	
	$f(x, y) = \frac{2x - y - 1}{\sqrt{2^2 + (-1)^2}}$	
	$f(x, y) = \frac{2x - y - 1}{\sqrt{5}}$	
8b	Soln.	
	$L_1 = 3x + 4y = 12$	
	$4y = -3x + 12$	
	$y = \frac{-3}{4}x + 3$	
	compare with $y = mx + c$.	
	$M_1 = -3/4$	
	and	
	$4x - 3y = 6 = L_2$	
	$3y = 4x - 6$.	
	$y = \frac{4}{3}x - 2$	
	compare with $y = mx + c$.	
	$M_2 = 4/3$	
	from	
	$\tan\theta = \frac{M_2 - M_1}{1 + M_1M_2}$	
	$\theta = \tan^{-1}\left(\frac{M_2 - M_1}{1 + m_1m_2}\right)$	

9b

$$\theta = \tan^{-1}\left(\frac{m_2 - m_1}{1 + m_1 m_2}\right)$$

$$\theta = \tan^{-1}\left(\frac{4/3 + +3/4}{1 + 4/3 \cdot -3/4}\right)$$

$$\theta = \tan^{-1}\left(\frac{25/12}{1 - 1}\right)$$

$$\theta = \tan^{-1}\left(\frac{25/12}{0}\right)$$

$$\theta = \tan^{-1}(\infty)$$

$$\theta = ~~0°~~$$

$$\theta = ~~45°~~ \; 90°$$

∴ the value of angles is ~~45°~~ 90°

8 (c) Solution:

Given:

$x^2+y^2+2x+2y-7=0$.

Point $(2,3)$.

Let the equation of tangent is

$y = mx+c$.

$mx - y + c = 0$

$mx - y + c = 0$

from:

$d = \left| \frac{ax+by+c}{\sqrt{a^2+b^2}} \right|$

$d = \left| \frac{ax+by+c}{\sqrt{a^2+b^2}} \right|$.

$d = \left| \frac{mx-y+c}{\sqrt{m^2+1}} \right|$

Point $(x,y) = (2,3)$

$d = \frac{2m-3+c}{\sqrt{m^2+1}}$ —(i)

from the equation of circle

$x^2+y^2+2x+2y-7=0$

Center $= (a,b) = (-1,-1)$

$c = -7$

radius $= \sqrt{a^2+b^2-c}$

Extract 8.2: A sample of incorrect responses to question 8 in paper 1

In Extract 8.2, part (a), the candidate failed to substitute the values of x and y, and hence failed to calculate the perpendicular distance of a point from a line. In part (b), he/she could not determine the equations of the angle bisectors; instead, he/she found the angle between the lines. In part (c), the candidate wrongly attempted to find the equation of the tangent and used it to calculate the length of the tangent from a point to a circle.

2.1.9 Question 9: Integration

The question examined the candidates' competence in techniques of integration of definite integrals and the application of integration to find the area under a curve and the volume of a solid of revolution about the x-axis. The question had the following parts:

(a) *Evaluate the definite integral* $\int_1^2 \frac{4x^2+3x-2}{(x+1)(2x+3)}dx$ *correct to four decimal places.*

(b) *Find the area enclosed between the curve* $y = x(x-1)(x-2)$ *and the x-axis.*

(c) *Find the volume of the solid formed by revolving the region enclosed by the curve* $y = x^2 - 4$ *and the* x*-axis about the* x*-axis by* $360°$.

The data shows that out of 18,136 (100%) candidates, 14,190 (78.24%) scored 0 to 3.0 marks, 2,335 (12.87%) scored 3.5 to 5.5 marks, and 1,611 (8.89%) scored 6.0 to 10 marks. Therefore, the candidates' performance on this question was generally weak. Figure 10 summarises the overall performance of candidates on this question.

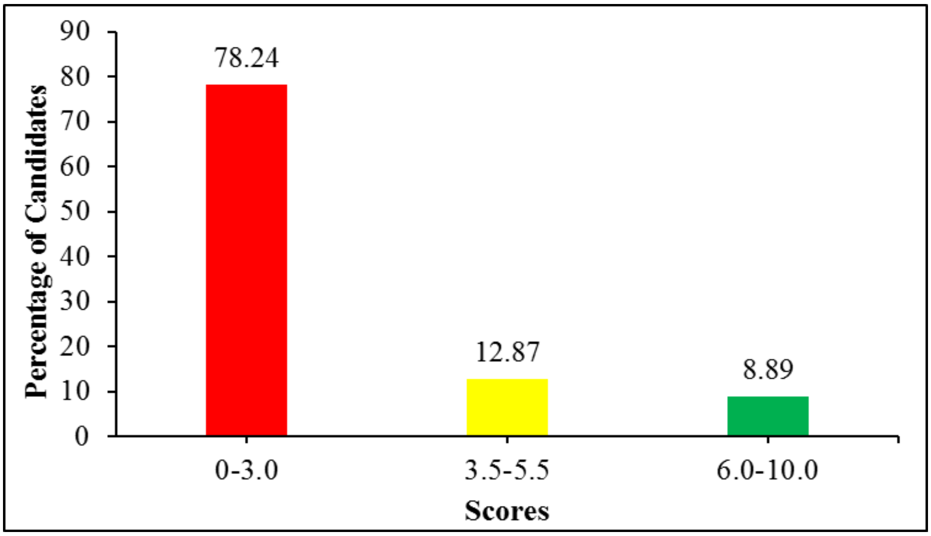


Figure 10: *The Candidates' Performance in Question 9 of Paper 1*

The data analysis further reveals that 7,176 (39.57%) candidates scored zero, indicating a lack of sufficient competence in the concept of integration and its application. They also had inadequate knowledge of integration techniques. In part (a), most candidates failed to decompose the rational function into partial fractions. They did not recognise that the given rational function was an improper fraction, which needed to be simplified through polynomial division. For example, they wrote $\frac{4x^2+3x+2}{(x+1)(2x+3)}=\frac{A}{x+1}+\frac{B}{2x+3}$, which led to incorrect partial fractions, $\frac{-5}{x+1}+\frac{13}{2x+3}$. Based on these, they evaluated the wrong integral, $\int_1^2\left(\frac{-5}{x+1}+\frac{13}{2x+3}\right)dx$, and ended up with incorrect answers. Other candidates applied an incorrect concept when manipulating the given rational function. For instance, they split the numerator into two parts by writing $numerator = A\frac{d}{dx}(deno\min ator)+B$, from which they obtained $4x^2+3x-2=\frac{1}{6}(6x+5)-\frac{9}{2}$, and the resulting integral became $\int_1^2\frac{\frac{1}{2}(6x+5)-\frac{9}{2}}{2x^2+5x+3}dx$, which could not yield the correct answer.

In part (b), the candidates incorrectly applied the limits of integration. They wrote $A=\int_0^2 x(x-1)(x-2)dx$ instead of $A=\int_0^1 x(x-1)(x-2)dx+\int_1^2 x(x-1)(x-2)dx$, which led them to an incorrect answer; $A=0$. Others failed to determine the limits of integration altogether. For example, they evaluated the indefinite integral $\int(x^3-3x^2+2x)dx$ and gave $\frac{1}{4}x^4-x^3+x^2+c$ as their final answer. In part (c), the candidates failed to interpret the question correctly and could not recall the correct formula for the volume of a solid of revolution. They wrote $V=\int_a^b \pi y dx=\int_{-2}^2 2\pi(x^2-4)dx$, which led to an incorrect answer: $V=67.021$. Extract 9.1 shows a sample of incorrect responses from candidates who answered the question incorrectly.

9. a) from

$$\int \frac{4x^2+3x-2}{(x+1)(2x+3)} dx.$$

Partialize the given.

$$\frac{4x^2+3x-2}{(x+1)(2x+3)} = \frac{A}{x+1} + \frac{B}{2x+3}$$

$$4x^2+3x-2 = A(2x+3) + B(x+1)$$

$$4x^2+3x-2 = 2xA + 3A + Bx + B.$$

So

$3 = 2A + B$ ...(i)

$-2 = 3A + B$

$A = -5$ $B = 13$

$$\text{So} \int \left(\frac{4x^2+3x-2}{(x+1)(2x+3)}\right) dx = \int \left(\frac{-5}{x+1} + \frac{13}{2x+3}\right) dx.$$

$$= \int \frac{13}{2x+3} dx + \int \frac{5}{x+1} dx.$$

for $\int \frac{1}{2x+3} dx.$

let $u = 2x+3.$

$dx = \frac{du}{2}$

$$13 \int \frac{1}{u} \cdot \frac{du}{2}$$

$$\frac{1}{2} \ln u.$$

$$\int \frac{13}{2x+3}\,dx = \frac{13}{2}\ln(2x+3)$$

and

$$\int \frac{5}{x+1}\,dx = 5\ln x+1$$

$$\int_1^2 \left(\frac{4x^2+3x-2}{(x+1)(2x+3)}\right)dx = \left[\frac{13}{2}\ln(2x+3) - 5\ln(x+1)\right]_1^2$$

$$= \left(\frac{13}{2}\ln 7 - 5\ln 3\right) - \left(\frac{13}{2}\ln 5 - 5\ln 2\right)$$

$$\frac{13}{2}\ln 7 - \frac{13}{2}\ln 5 - 5\ln 3 + 5\ln 2.$$

$$\int_1^2 \frac{4x^2+3x-2}{(x+1)(2x+3)} = \frac{13}{2}\ln 7/5 + 5\ln 2/3$$

$$= 0.78314.$$

$$\therefore \int_1^2 \frac{4x^2+3x-2}{(x+1)(2x+3)} = 0.78314$$

9 (c) Solution

$$\text{Volume} = \int \pi y\,dx$$

$$y = x^2 - 4$$

$$x^2 = 4$$

$$x = 2,\ x = -2.$$

Thus:

$$\text{Volume} = \int_{-2}^{2} \pi (x^2-4)\,dx$$

$$= \frac{32}{3}\pi \text{ cubic unit}$$

$$\therefore \text{The Volume} = \frac{32}{3}\pi \text{ cubic}$$

Extract 9.1: A sample of incorrect responses to question 9 of paper 1

In Extract 9.1, part (a), the candidate could not recognise that the given rational function was an improper fraction that needed to be divided before being partialised. Also, in part (c), he/she failed to recall the correct formula for the volume of a solid of revolution.

Despite the weak performance of most candidates, the analysis of responses shows that 185 (1.02%) candidates scored full marks. These candidates demonstrated good competence in the concept of integration. They applied appropriate techniques of integration to answer the question. In part (a), the candidates decomposed the rational function $\frac{4x^2+3x-2}{(x+1)(2x+3)}$ into its equivalent partial fractions $2-\frac{1}{x+1}-\frac{5}{2x+3}$. Finally, they correctly evaluated the definite integral $\int_1^2\left(2-\frac{1}{x+1}-\frac{5}{2x+3}\right)dx$ to obtain the correct answer 0.7534. In part (b), the candidates correctly applied the formula for calculating the area under the curve. First, they found the limits of integration by equating the function $y=x(x-1)(x-2)$ to zero. They recognised that the area under the curve lies both below and above the x-axis. They applied the correct formula for the area under the curve $A=\int_a^b ydx+\int_b^c ydx$ and substituted the values into it to obtain the definite integral $A=\int_0^1\left(x(x-1)(x-2)\right)dx+\int_1^2\left(-x(x-1)(x-2)\right)dx$. Finally, they evaluated the integral and obtained the correct answer of $A=\frac{1}{2}$ square units.

In part (c), the candidates recalled the correct formula for the volume of a solid of revolution about the x-axis, that is, $V=\pi\int_a^b y^2dx$. They equated the given function $y=x^2-4$ to zero to obtain the limits of integration, $x=-2$ and $x=2$. Upon substituting the function y and the limits into the formula, they obtained the integral $V = \pi \int_{-2}^{2} \left(x^2 - 4\right)^2 dx$. After evaluating this integral, they arrived at the correct answer of $V = \frac{512}{15}\pi$ cubic units. Extract 9.2 provides a sample of correct responses from one of the candidates who had adequate skills in attempting this question.

9\. a) Consider:

$$\frac{4x^2 + 3x - 2}{2x^2 + 3x + 2x + 3}$$

$$\frac{4x^2 + 3x - 2}{2x^2 + 5x + 3}$$

Upon long dividing

$$\begin{array}{r|l} & \;\; 2 \\ 2x^2 + 5x + 3 & 4x^2 + 3x - 2 \\ & -\;4x^2 + 10x + 6 \\ & \qquad -7x - 8 \end{array}$$

$$2 + \frac{-7x - 8}{(x+1)(2x+3)}$$

Partializing

$$\frac{-7x - 8}{(x+1)(2x+3)} = \frac{A}{x+1} + \frac{B}{2x+3}$$

$$\frac{-7x - 8}{(x+1)(2x+3)} = \frac{A(2x+3) + B(x+1)}{(x+1)(2x+3)}$$

$$-7x - 8 = A(2x+3) + B(x+1)$$

when $x = -1$.

$$-7(-1) - 8 = A(-2+3) + 0$$

$$7 - 8 = A$$

$$A = -1$$

when $x = 0$

$$-8 = 3(-1) + B$$

$$-8 = -3 + B$$

$$B = -5$$

9.

$$= \frac{-1}{x+1} - \frac{5}{2x+3}$$

$$\int \frac{4x^2+3x-2}{(x+1)(2x+3)}\,dx = \int 2\,dx - \int \frac{1}{x+1}\,dx - \int \frac{5}{2x+3}\,dx$$

$$= 2x - \ln(x+1) - \tfrac{5}{2}\ln(2x+3) + c$$

Using limits 1 —— 2

$$= \left[(2\times 2) - \ln(2+1) - \tfrac{5}{2}\ln(2\times 2+3)\right] - \text{when } x = 1$$

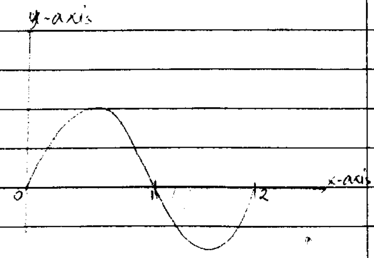
$$= -1.96339 - -2.71674$$

$$= 0.75335$$

$$\therefore \int_1^2 \frac{4x^2+3x-2}{(x+1)(2x+3)}\,dx = 0.7534$$

9b solution.

$$\text{Area} = \int_a^b y\,dx.$$



but, x-axis, $y = 0$

$$y = x(x-1)(x-2)$$

$$0 = x(x-1)(x-2)$$

$x = 0$, $x = 1$, $x = 2$.

$$\text{Area} = \int_0^1 [x(x-1)(x-2)]\,dx + \left|\int_1^2 [x(x-1)(x-2)]\,dx\right|$$

9b. $A = \int_0^1 (x(x^3 - 3x + 2))\,dx + \left| \int_1^2 (x^3 - 3x^2 + 2x)\,dx \right|$

$A = \left. \frac{x^4}{4} - \frac{3x^3}{3} + \frac{2x^2}{2} \right|_0^1 + \left| \left. \frac{x^4}{4} - x^3 + x^2 \right|_1^2 \right|$

$A = \left(\frac{1}{4} - 1 + 1 \right) + \left| \left(\frac{(2)^4}{4} - (2)^3 + (2)^2 - \frac{1}{4} \right) \right|$

$A = \frac{1}{4} + \left| -\frac{1}{4} \right|$

$A = \frac{1}{4} + \frac{1}{4}$

$A = \frac{1}{2}$ sq units.

$\therefore$ The area of the curve is $\frac{1}{2}$ sq units.

9c) solution.

for volume

$V = \pi \int_a^b (y_2 - y_1)^2\,dx$

$y_1 = 0$, $y_2 = x^2 - 4$

$0 = x^2 - 4$

$x^2 = 4$

$x = \pm 2$.

$V = \pi \int_{-2}^{2} (x^2 - 4 - 0)^2\,dx$.

$$V = \pi \int_{-2}^{2} (x^4 - 8x^3 + 16)\, dx .$$

$$V = \pi \left(\frac{x^5}{5} - \frac{8x^3}{3} + 16x \right) \Big|_{-2}^{2}$$

$$V = \pi \left[\left(\frac{(2)^5}{5} - \frac{8(2)^3}{3} + 16(2) \right) - \left(\frac{(-2)^5}{5} - \frac{8(-2)^3}{3} + 16(-2) \right) \right]$$

$$V = \pi \left[\frac{256}{15} + \frac{256}{15} \right]$$

$$V = \pi \left(\frac{512}{15} \right)$$

$$V = \frac{512\pi}{15} \text{ cubic units}$$

$\therefore$ The volume is $\frac{512\pi}{15}$ cubic units.

Extract 9.2: A sample of correct responses to question 9 of paper 1

In Extract 9.2, the candidate applied appropriate techniques of integration to evaluate the given integrals, find the area under the curve, and determine the volume of the solid of revolution about the x-axis.

2.1.10 Question 10: Differentiation

The question assessed the candidates' ability to apply differentiation techniques to solve various problems. The question was stated as follows:

(a) Differentiate $x^3 y + y^3 x = 2y$ with respect to x at (x, y).

(b) If a car starts from rest and moves a distance of g cm in t seconds where $g = \frac{1}{8}t^4 + \frac{1}{2}t^2$,

(i) find the velocity of the car after two seconds.

(ii) find the initial acceleration.

(c) Differentiate the following expressions with respect to x:

(i) $\dfrac{e^{x^3}(\sin x)^{\frac{1}{2}}}{3x+1}$

(ii) $\cos^{-1}(\tan x)$

The data analysis shows that, out of 18,136 (100%) candidates, 6,460 (35.62%) scored 0 to 3.0 marks, 3,736 (20.60%) scored 3.5 to 5.5 marks, and 7,940 (43.78%) candidates scored 6.0 to 10 marks. Figure 11 summarizes the overall performance of candidates on this question.

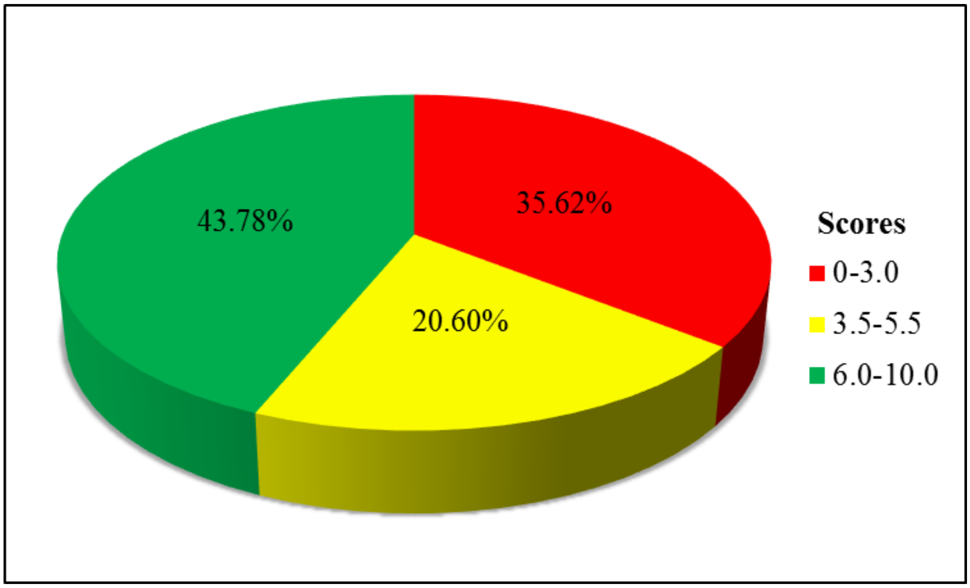


Figure 11: *The Candidates' Performance in Question 10 of Paper 1*

As Figure 11 shows, 11,676 (64.38%) candidates scored 3.5 to 10 marks, indicating good performance on this question. An analysis of the responses shows that the candidates who performed well in part (a) applied implicit differentiation to evaluate the derivative of the function at $(1,1)$. They differentiated $x^3y + y^3x = 2y$ to obtain $\frac{dy}{dx} = \frac{3x^2y + y^3}{2 - x^3 - 3y^2x}$, and then substituted $(1,1)$ to get the correct answer, $\frac{dy}{dx} = -2$. In part (b), the candidates recognised that velocity is the derivative of displacement, and acceleration is the derivative of velocity. Thus, in part (b) (i), they differentiated the displacement function $g = \frac{1}{8}t^4 + \frac{1}{2}t^2$ to obtain the velocity at any time, $v = \frac{dg}{dt} = \frac{1}{2}t^3 + t$. Upon substituting $t = 2$ seconds, they obtained the correct answer, $v = 6$ cm/s. In part (b) (ii), they differentiated the velocity function $v = \frac{1}{2}t^3 + t$ to obtain the acceleration $a = \frac{dv}{dt} = \frac{3}{2}t^2 + 1$. Finally, they substituted $t = 0$ to get the initial acceleration, $a = 1 \text{ cm/s}^2$.

In part (c), the candidates applied proper techniques to differentiate the given expressions. In part (c) (i), they introduced natural logarithm, that is $\ln y = \ln\left(\frac{e^{x^3}(\sin x)^{\frac{1}{2}}}{3x+1}\right)$. Then, they applied appropriate laws of logarithms to determine the derivative of the resulting function with respect to x and obtained $\frac{dy}{dx} = \left(\frac{e^{x^3}(\sin x)^{\frac{1}{2}}}{3x+1}\right)\left(3x^2 + \frac{\cos x}{2\sin x} - \frac{3}{(3x+1)}\right)$. Moreover, others differentiated the expression by applying the quotient rule as follows: $\frac{dy}{dx} = \frac{(3x+1)\left(e^{x^3}(\sin x)^{\frac{1}{2}}\right)' - \left(e^{x^3}(\sin x)^{\frac{1}{2}}\right)(3x+1)'}{(3x+1)^2}$. On simplifying, they ended up with an equivalent function of the derivative, that is $\frac{dy}{dx} = \frac{(3x+1)\left(e^{x^3}\left(\frac{\cos x}{2\sqrt{\sin x}}\right) + \left((\sin x)^{\frac{1}{2}}\right)\left(3x^2 e^{x^3}\right)\right) - \left(e^{x^3}(\sin x)^{\frac{1}{2}}\right)(3)}{(3x+1)^2}$. In part (c) (ii), the candidates transformed the given expression $\cos^{-1}(\tan x)$ into $\cos y = \tan x$ and then differentiated the equation to obtain $\sec^2 x = -\sin y \frac{dy}{dx}$. Then, they simplified the equation to obtain the correct derivative, $\frac{dy}{dx} = -\frac{(1+\tan^2 x)}{\sqrt{1-\tan^2 x}}$. Extract 10.1 is a sample of correct responses from one of the candidates who attempted this question correctly.

10. a/ Soln:

given, $x^3y + y^3x = 2y$

$$x^3\frac{dy}{dx} + 3x^2y + 3y^2x\frac{dy}{dx} + y^3 = 2\frac{dy}{dx}$$

$$\frac{dy}{dx}(x^3 + 3y^2x - 2) = -(y^3 + 3x^2y)$$

$$\frac{dy}{dx} = \frac{-(y^3 + 3x^2y)}{x^3 + 3y^2x - 2}$$

$$\frac{dy}{dx} = \frac{-(1+3)}{1 + 1(3) - 2}$$

$$\frac{dy}{dx} = \frac{-4}{2} = -2$$

$\therefore \frac{dy}{dx}$ at $(1,1) = -2$

b/ Soln:

i/ from, $\frac{dg}{dt} = \frac{4t^3}{8} + t$ at $t = 2$

$$\frac{dg}{dt} = \frac{4(2^3)}{8} + 2$$

$$\frac{dg}{dt} = 4 + 2 = 6 \text{ cm/sec}$$

$\therefore$ Velocity after 2 sec is $V = 6$ cm/sec

ii/ Soln:

from, $\frac{d^2g}{dt^2} = \frac{12t^2}{8} + 1$

10 b) $\frac{d^2g}{dt^2} = \frac{3t^2}{2} + 1$ at (0,0).

$$\frac{d^2g}{dt^2} = \frac{3(0)^2}{2} + 1$$

$$\frac{d^2g}{dt^2} = 1 \text{ cm/sec}^2$$

∴ Initial acceleration, $\frac{d^2g}{dt^2} = 1 \text{ cm/sec}^2$

c) Soln.

i) Let $y = \frac{e^{x^3}(\sin x)^{1/2}}{3x+1}$

By applying ln

$$\ln y = \ln e^{x^3} + \frac{1}{2}\ln(\sin x) - \ln(3x+1)$$

$$\ln y = x^3 + \frac{1}{2}\ln(\sin x) - \ln(3x+1)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = 3x^2 + \frac{1}{2}\frac{\cos x}{\sin x} - \frac{3}{3x+1}$$

$$\frac{dy}{dx} = \left[3x^2 + \frac{1}{2}\cot x - \frac{3}{3x+1}\right]y$$

but $y = \frac{e^{x^3}(\sin x)^{1/2}}{3x+1}$

$$\therefore \frac{dy}{dx} = \left[3x^2 + \frac{1}{2}\cot x - \frac{3}{3x+1}\right]\frac{e^{x^3}(\sin x)^{1/2}}{3x+1}$$

10. c) ii) Soln.

given $y = \cos^{-1}(\tan x)$

let $u = \tan x$

$du = \sec^2 x \, dx$

$y = \cos^{-1}(u)$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}} \cdot \sec^2 x$$

but $u = \tan x$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\tan^2 x}} \cdot \sec^2 x$$

$$\frac{dy}{dx} = \frac{-\sec^2 x}{\sqrt{1-\tan^2 x}}$$

$$\therefore \frac{dy}{dx} = \frac{-\sec^2 x}{\sqrt{1-\tan^2 x}}$$

Extract 10.1: A sample of correct responses to question 10 of paper 1

In Extract 10.1, in part (a), the candidate evaluated the derivative of the function at the given point after differentiating it correctly. In part (b), he/she applied the concept of differentiation to find the velocity and acceleration correctly. In part (c), the candidate applied suitable techniques of differentiation to find the derivatives of the given expressions.

On the other hand, 2,871 (15.83%) candidates scored zero, and an analysis of their responses indicated that they failed to apply differentiation techniques correctly. In part (a), some candidates confused the concept of implicit differentiation with partial differentiation. Using the concept of partial derivatives, they differentiated the equation $x^3y + y^3x = 2y$ to obtain $f_x(x, y) = 3yx^2 + 3xy^2$, which led them to an incorrect answer, $f_x(1, 1) = 6$. In part (b), the candidates failed to differentiate the expression for displacement $g=\frac{1}{8}t^4+\frac{1}{2}t^2$, to find the velocity and acceleration. Additionally, some failed to recognise that the initial acceleration is obtained when $t=0$. They substituted an incorrect value of $t=2$ seconds to find the initial acceleration. In part (c) (i), the candidates failed to apply the correct techniques of differentiation to find the derivative of the given expression. They wrote $y=\frac{e^{x^3}(\sin x)^{\frac{1}{2}}}{3x+1}=e^{x^3}(\sin x)^{\frac{1}{2}}(3x+1)^{-1}$ and then $\frac{dy}{dx}=3e^{x^2}\frac{1}{2\sqrt{\cos x}}.(-1)(3)^{-1}$, which is incorrect. In part (c) (ii), they recalled an incorrect formula for the derivative of $\cos^{-1}(x)$ and also failed to apply the concept of the derivative of a composite function. Extract 10.2 shows a sample of incorrect responses from candidates who answered the question incorrectly.

10.	a) $x^3y + y^3x = 2y$	
	$3x^2y + y^3 = 0$ at $(x,y)=(1,1)$	
	$3(1^2)(1) + (1)^3 = 0$	
	$= 3.$	
	$\therefore = 3.$	

10. solution.

(b) $y = \frac{1}{8}t^4 + \frac{1}{2}t^2$

Velocity $= \frac{1}{2}t^3 + \frac{1}{4}t$

$t = 2$

$= \frac{1}{2}(2)^3 + \frac{1}{4}(2)$

$= 4.5$ cm/sec

acceleration

$\frac{dy}{dx} = \frac{1}{2}t^3 + \frac{1}{4}t$

$= \frac{3}{2}t^2 + \frac{1}{4}$

$= 6.25$ cm/sec

Initial acceleration is 6.25 cm/sec

(c) $\cos^{-1}(\tan x)$

Let $\tan x = u$

$y = \cos^{-1} u$

$\cos y = u$

i. soln

$y = \frac{e^{x^3}(\sin x)^{1/2}}{3x+1}$

$b = e^{x^3}(\sin x)^{1/2} \cdot (3x+1)^{-1}$.

$\frac{dy}{dx} = 3ex^2 \cdot \frac{1}{2\sqrt{\cos x}} \cdot -1(3)^{-2}$

$\frac{dy}{dx} = \frac{3ex^2 - 3^{-2}}{2\sqrt{\cos x}}$

$\frac{dy}{dx}$

Extract 10.2: A sample of incorrect responses to question 10 of paper 1

In Extract 10.2, part (a), the candidate substituted $(x, y) = (1, 1)$ into the equation instead of differentiating it using the concept of implicit differentiation. In part (b), the candidate failed to differentiate correctly and did not recognise that the initial acceleration is obtained when $t = 0$. In part (c), the candidate incorrectly applied differentiation techniques to find the derivatives of the given expressions.

2.2 142/2 ADVANCED MATHEMATICS 2

2.2.1 Question 1: Probability

The question examined the candidates' understanding of the concepts of probability density and normal distribution. The question had two parts, (a) and (b), which read as follows:

(a) A function is defined as $f(x) = \frac{3}{26}x^2$*;*

(i) Show that this function is a probability density function of a random variable X for the interval $1 \le x \le 3$.

(ii) Find the mean and standard deviation of X for the interval given in part (a) (i).

(b) The mean weight and standard deviation for 2500 students at Kabarimu Primary School as recorded in 1989 were 35 kilograms and 6.3 kilograms respectively. Assuming that their weights are normally distributed, determine;

(i) the probability that a student picked at random will weigh more than 32 kilograms.

(ii) the number of overweight students at this school, if an overweight is considered for all students weighing above 55 kilograms.

A total of 18,134 (100%) candidates responded to this question, of whom 6,335 (34.93%) scored from 0 to 5.0 marks, while 11,799 (65.07%) scored from 5.5 marks and above. Thus, the overall performance of the candidates on this question was good. Figure 12 summarizes the candidates' performance on this question.

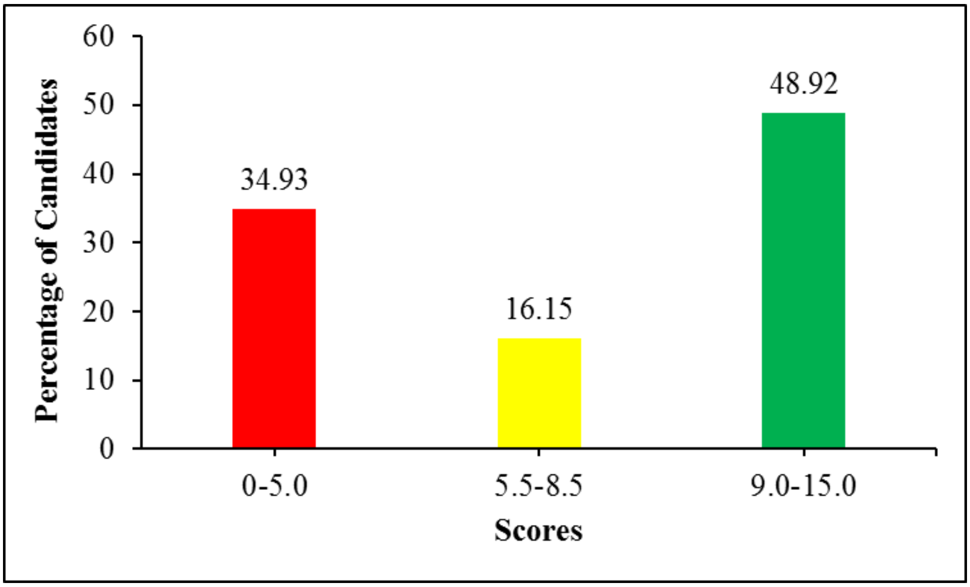


Figure 12: *The Candidates' Performance in Question 1 of Paper 2*

The data further reveals that 1,500 (8.27%) candidates scored full marks, indicating that they had sufficient knowledge of the properties of a probability density function and the concept of the normal distribution. In part (a) (i), the candidates applied the correct property to show that the given function is a probability density function. They wrote: $f(x) = \frac{1}{3}x^2$ for the function to be a probability density function. Then, they evaluated $\int_1^3 \frac{3}{26}x^2 dx$ and obtained 1, which confirms that the given function is a probability density function. In part (a) (ii), the candidates calculated the mean and standard deviation by applying the correct formulae. To calculate the mean, they evaluated $E(X) = \int_1^3 xf(x)dx$, that is, $E(X) = \int_1^3 x\left(\frac{3x^2}{26}\right)dx = \int_1^3 \left(\frac{3x^3}{26}\right)dx$, and obtained $E(X) = 2.31$. To calculate the standard deviation, they evaluated $\delta = \sqrt{E(X^2) - [E(X)]^2}$, whereby $E(X^2) = \int_1^3 x^2 f(x)dx$; hence, $E(x^2) = 5.5846$. Thereafter, they obtained the correct answer of $\delta = 0.4985$. In part (b) (i), the candidates calculated the probability $P(X > 32)$. They converted the normal random variable X to the standardised random variable Z by writing $P(X > 32) = P\left(Z > \frac{32-\mu}{\sigma}\right)$, and upon evaluation, they obtained $P(X > 32) = 0.68303$. In part (b) (ii), the candidates calculated the number of overweight students by using $N = P(X > 55) \times n(S)$, where $P(X > 55) = 0.0007$ and $n(S) = 2500$; hence, they obtained $N = 2$. Extract 11.1 is a sample of correct responses from one of the candidates who answered the question correctly.

1. a) Soln:

For a probability density function:

$$\int_a^b f(x)\,dx = 1$$

From, $f(x) = \frac{3}{26}x^2$

$$\int_a^b f(x)\,dx = \int_1^3 \frac{3}{26}x^2\,dx$$

$$= \frac{3}{26}\int_1^3 x^2\,dx$$

$$= \frac{3}{26}\left.\frac{x^3}{3}\right|_1^3$$

$$= \left.\frac{x^3}{26}\right|_1^3$$

$$= \frac{(3)^3}{26} - \frac{1^3}{26}$$

$$= \frac{27}{26} - \frac{1}{26}$$

$$= \frac{27-1}{26} = \frac{26}{26} = 1$$

$$\int_a^b f(x)\,dx = \int_1^3 \frac{3}{26}x^2\,dx = 1$$ hence shown.

1 a) ii Soln.

For

$$\text{Mean} = \int_a^b x f(x)\, dx$$

$$\int_a^b x f(x)\, dx = \int_1^3 \left(\frac{3}{26} x^2\right) x \, dx$$

$$= \frac{3}{26} \int_1^3 x^3 \, dx$$

$$= \frac{3}{26} \left[\frac{x^4}{4}\right]_1^3$$

$$= \frac{3}{26} \left[\frac{3^4}{4} - \frac{1^4}{4}\right]$$

$$= \frac{3}{26} \left[\frac{81}{4} - \frac{1}{4}\right]$$

$$= \frac{3}{26} \times \frac{80}{4}$$

$$\underline{\text{Mean}} = \frac{30}{13}$$

then

$$S.D = \sqrt{\text{variance}}$$

$$\text{Variance}(x) = E(x^2) - \left(E(x)\right)^2$$

$$E(x^2) = \int_a^b x^2 f(x)\, dx$$

1 a) ii

$$E(x^2) = \int_1^3 x^2\left(\frac{3x^2}{26}\right)dx$$

$$= \frac{3}{26}\int_1^3 x^4 dx$$

$$= \frac{3}{26}\left[\frac{x^5}{5}\right]_1^3$$

$$= \frac{3}{26}\left[\frac{3^5}{5} - \frac{1^5}{5}\right]$$

$$= \frac{3}{26}\left[\frac{243}{5} - \frac{1}{5}\right]$$

$$= \frac{3}{26}\left[\frac{242}{5}\right]$$

$$= \frac{363}{65}$$

Then,

$$Var(x) = \frac{363}{65} - \left(\frac{30}{13}\right)^2$$

$$Var(x) = \frac{363}{65} - \frac{900}{169}$$

$$Var(x) = \frac{219}{845}$$

$$S.D = \sqrt{Var(x)} = \sqrt{\frac{219}{845}} = 0.5091$$

$$\therefore \text{Mean}(x) = \frac{30}{31}$$

Standard deviation = 0.5091

1 b) Soln.

given, $n = 2500$

$\mu, \bar{x} = 35$ kg

$\delta = 6.3$ kg

i) Required, $P(x > 32)$.

From,

$$P(x) = P\left(z > \frac{x-\mu}{\delta}\right).$$

$$P(x>32) = P\left(z > \frac{32-35}{6.3}\right)$$

$$P(x>32) = P\left(z > \frac{-10}{21}\right)$$

$$P(x>32) = 0.68303$$

∴ the probability, $P(x > 32) = 0.68303$

ii) Required $n(x>55)$.

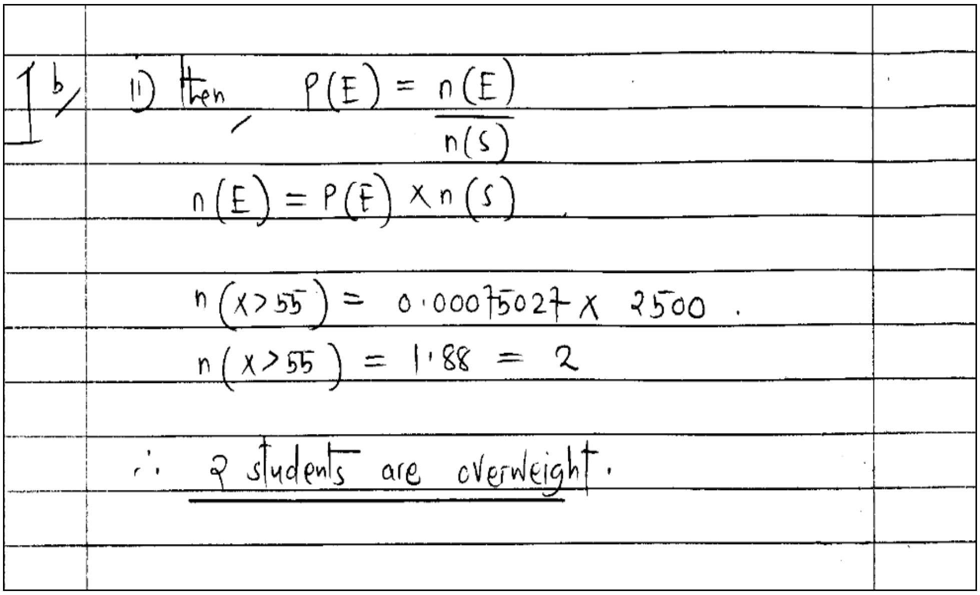
From,

$$P(x) = P\left(z > \frac{x-\mu}{\delta}\right)$$

$$P(x>55) = P\left(z > \frac{55-35}{6.3}\right)$$

$$P(x>55) = P\left(z > \frac{200}{63}\right)$$

$$P(x>55) = 0.00075027$$


1 b/ ii) then $P(E) = \frac{n(E)}{n(S)}$

$n(E) = P(E) \times n(S)$

$n(X>55) = 0.00075027 \times 2500$.

$n(X>55) = 1.88 = 2$

∴ 2 students are overweight.

Extract 11.1: A sample of correct responses to question 1 of paper 2

In Extract 11.1, part (a) (i), the candidate verified that the given function is a probability density function by showing that $\int_1^3 f(x)dx = 1$. In part (a) (ii), the candidate calculated the mean and standard deviation by applying the correct formulae. In part (b) (i), the candidate calculated the required probability using $P(X > 32) = P\left(Z > \frac{32-\mu}{\sigma}\right)$. In part (b) (ii), the candidate correctly calculated the number of overweight students using $N = P(X > 55) \times n(S)$.

Further analysis of the data shows that, among the 6,335 (34.93%) candidates who scored less than 5.5 marks, 2,769 (15.27%) scored zero. This indicates that the candidates who performed poorly on this question had insufficient knowledge of the properties of a probability density function and the concept of the normal distribution. In part (a) (i), the candidates failed to verify that the given function is a probability density function. For instance, some candidates wrote an irrelevant formula $P(x = x) = \int_a^b xf(x)dx$, and ended up with an incorrect conclusion. In part (a) (ii), the candidates applied incorrect concepts to find the mean and standard deviation. Most of them used formulae for discrete random variables instead of those for continuous random variables. For example, they wrote $E(X)=\sum xP(x)$ and prepared a probability distribution table for the values of x and $P(x)$, which led to incorrect results.

In part (b), the candidates confused the given normal distribution with a discrete probability distribution that requires approximation using the normal distribution. They incorrectly applied the continuity correction of 0.5, which is used to convert a discrete random variable into a continuous one. They also confused the number of overweight students with the probability of overweight students. Extract 11.2 presents a sample of incorrect responses from one of the candidates who answered the question incorrectly.

1a) solution

Given

$f(x) = \frac{3}{26}x^2$

from

$P(x=x) = \int_a^b x f(x)\,dx$

$P(X=u) = \int_1^3 x \frac{3}{26}x^2\,dx$

$= \frac{3}{26}\int_1^3 x^3\,dx$

$= \frac{3}{26}\left[\frac{x^4}{4}\right]_1^3$

$= \frac{3}{26} \times 20$

$= \frac{30}{13}$

$\therefore P(x) = \frac{30}{13}$

b) solution

Mean $E(x) = \sum x\,p(x)$

but

Mean $(\bar{x}) = \sum x f(x)$

x	1	2	3		
f(x)	3/26	6/13	27/26		

$\bar{x} = 1 \times 3/26 + 2 \times 6/13 + 3 \times 27/16$

$\bar{x} = 1269/208$

$s \cdot d = \sqrt{E(x)^2 - [E(x)]^2}$

$E(x)^2 = x \sum x p(x)$
$= \frac{35}{6}$
$= \frac{3567}{208}$

$s \cdot d = \sqrt{\frac{3567}{208} - \left(\frac{1269}{208}\right)^2}$

$s \cdot d = 11.07$

b) solution
Mean $(\bar{x}) =$ ~~17~~ 84 35
$s \cdot d = 6.3$
Total student = 2500
Value of $x = 32$

from $z = \frac{x - \mu}{s \cdot d}$

$$Z = \frac{(32 - 0.5) - 35 < (32 + 0.5) - 35}{6.3}$$

$$Z = \frac{-3.5 < 2.5}{6.3}$$

$$Z = -0.55 < -0.3960$$

$$Z = 0.36279$$

∴ The probability is 0.363.

11/ solution

pm

$$Z = \frac{X - \mu}{s.d}$$

$$Z = \frac{31.5 - 55 > 32.5 - 55}{6.3}$$

$$Z = -3.73 > -3.57$$

$$Z = 0.999$$

∴ The number of overweight = 0.999

Extract 11.2: A sample of incorrect responses to question 1 of paper 2

In Extract 11.2, part (a), the candidate failed to verify that the given function is a probability density function and also applied incorrect concepts of discrete random variables instead of those for continuous ones to find the mean and standard deviation. In part (b), the candidate confused the given normal distribution with a discrete probability distribution.

2.2.2 Question 2: Logic

The question assessed the candidate's understanding of the concepts of contrapositive, electrical networks, logical symbols and simplification, and argumentation. The question was stated as follows:

(a) *For each of the following statements, write down the corresponding contrapositive statement:*

(i) *If the graph of* $y = mx + b$ *is an oblique, then* $m \neq 0$.

(ii) *If a quadrilateral has two sides of equal lengths, then the quadrilateral has two equal angles.*

(iii) *If it rains today, then it will rain tomorrow.*

(iv) *If Ubaya committed a crime, then he will be at the crime scene.*

(b) *Draw an electrical network corresponding to the proposition* $(P \wedge \sim Q) \vee ((\sim P \vee R) \wedge Q)$.

(c) *Eliminate the logical connective symbol "*$\rightarrow$*" from the statement* $(\sim P \wedge \sim Q) \rightarrow (Q \rightarrow R)$ *and then simplify the statement as much as possible.*

(d) *Test the validity of the argument "If I like Mathematics, then I will study hard. Either I study hard, or I fail. Therefore, if I fail, then I do not like Mathematics."*

The data reveals that 18,134 (100%) candidates attempted this question. Among them, 17,333 (95.58%) scored 5.5 marks or more, and 801 (4.42%) scored from 0 to 5.0 marks. Therefore, the overall performance of the candidates on this question was good. Figure 13 summarises the candidates' performance on this question.

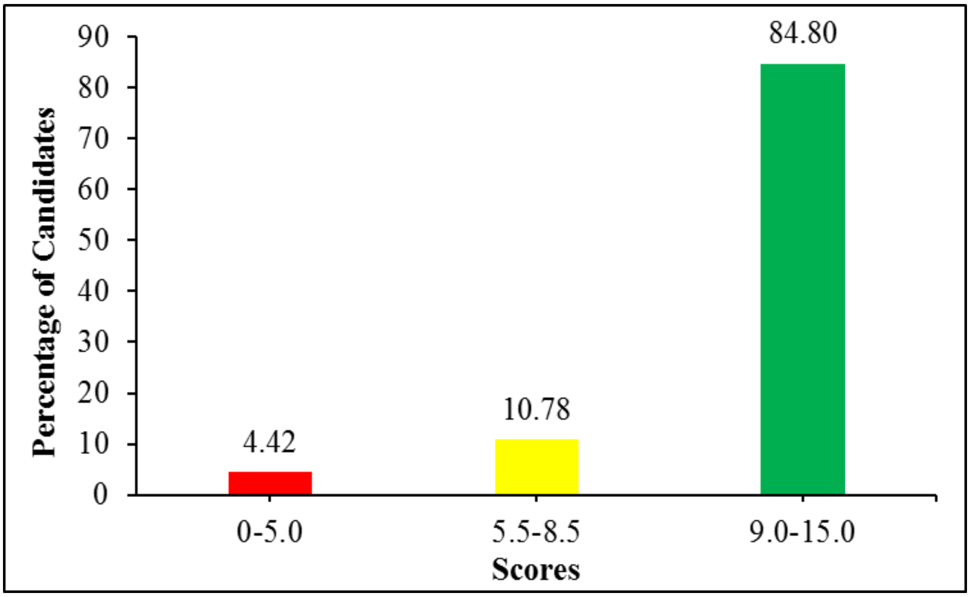


Figure 13: *The Candidates' Performance in Question 2 of Paper 2*

Analysis of responses shows that the candidates who performed well on this question had a good understanding of the concepts of contrapositive, electrical networks, logical symbols and simplification, and arguments. In part (a), the candidates wrote the correct contrapositive statements of the given propositions. In part (b), the candidates correctly drew an electrical network corresponding to the given proposition $(P \wedge \sim Q) \vee ((\sim P \vee R) \wedge Q)$. In part (c), the candidates eliminated "$\rightarrow$" from the statement $(\sim P \wedge \sim Q) \rightarrow (Q \rightarrow R)$ and simplified it to obtain the tautology T. In part (d), the candidates converted the given argument from verbal form to symbolic form $[(P \rightarrow Q) \wedge (Q \vee R)] \rightarrow (R \rightarrow \sim P)$ and then used a truth table to test its validity. From the truth table, they found the argument to be invalid. Extract 12.1 shows a correct response from a candidate who answered the question well.

2. a/

i) If $m = 0$, Then the graph of $y = mx + b$ is not an oblique.

ii) If the quadrilateral does not have two equal angles Then the quadrilateral does not have two sides of equal lengths.

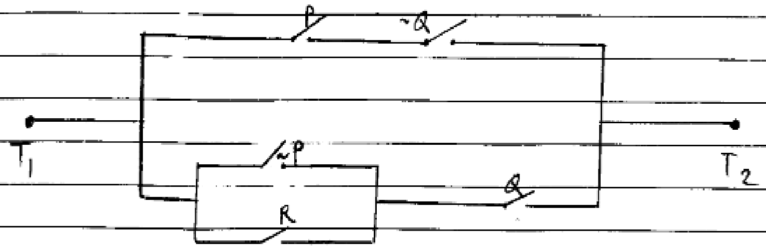
iii) If it does not rain tommorrow, Then it will not rain today.

iv) If Ubaya will not be at the crime scene, Then he did not commit a crime.

b) Network

$(p \wedge \sim q) \vee [(\sim p \vee R) \wedge Q]$

2 b)



Soln

c/ Given: $(\sim P \wedge \sim Q) \rightarrow (Q \rightarrow R)$.

$= (\sim P \wedge \sim Q) \rightarrow (Q \rightarrow R)$ — — — — — given

$= \sim(\sim P \wedge \sim Q) \vee (\sim Q \vee R)$ — — — — Implication law

$= (P \vee Q) \vee (\sim Q \vee R)$ — — — — De Morgans law

$= P \vee R \vee (Q \vee \sim Q)$ — — — — Associative law

$= P \vee R \vee T$ — — — — — — — Negation law

$= (P \vee R) \vee T$ — — — — — — Associative law

$= T$ — — — — — — — — Identity law

$\therefore (\sim P \wedge \sim Q) \rightarrow (Q \rightarrow R) \equiv T$

d/ Soln.

given, let p be I like Mathematics

q be I will study hard.

r be I Fail.

Statement: $[(p \rightarrow q) \wedge (q \vee r)] \rightarrow (r \rightarrow \sim p)$.

By Using truth table;

2. d/

				(A)	(B)	(C)
p	q	r	$\sim p$	$p \to q$	$q \vee r$	$A \wedge B$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	F	T	F
T	F	F	F	F	F	F
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T
F	F	F	T	T	F	F

$r \to \sim p$	$c \to (r \to \sim p)$
F	F
T	T
F	T
T	T
T	T
T	T
T	T
T	T

∴ Since the last column is not tautology The argument is not valid.

Extract 12.1: A sample of correct responses to question 2 of paper 2

In Extract 12.1, part (a), the candidate wrote the correct contrapositive statements, while in part (b), the candidate drew the correct electrical network. In part (c), the candidate eliminated "$\to$" from the given statement and simplified it correctly. In part (d), the candidate converted the given argument and then correctly tested its validity using the truth table.

Despite the good performance of most candidates on this question, 105 (0.58%) scored zero due to various challenges they faced**.** In part (a), the candidates failed to correctly write the contrapositive statements of the given propositions. In part (b), the candidates failed to draw an electrical network for the given proposition $(P \wedge \sim Q) \vee ((\sim P \vee R) \wedge Q)$. In part (c), the candidates failed to eliminate the logical connective symbol "$\to$" from the statement $(\sim P \wedge \sim Q) \to (Q \to R)$, as well as to use the laws to simplify the proposition. In part (d), the candidates failed to convert the given argument into symbolic form using logical connectives. Extract 12.2 is a sample of incorrect responses from one of the candidates who answered the question incorrectly.

2(a) i) let, p ⇒ The graph of $y = mx + b$ is an oblique.

$q = m \neq 0$

Statement: $p \rightarrow q$

Contrapositive: $\sim q \rightarrow \sim p$

∴ It is not true that if $m \neq 0$, then the graph of $y = mx + b$ is an oblique

ii). let, p = A quadrilateral has two sides of equal lengths.

q = the quadrilateral has two equal angles

Statement: $p \rightarrow q$.

Contrapositive: $\sim q \rightarrow \sim p$.

∴ It is false that if a quadrilateral has two equal angles, then a quadrilateral has two sides of equal lengths.

iii) let, p = It rains today.

q = It will rain tomorrow

2 (a) Statement : $p \rightarrow q$

Contrapositive : $\sim q \rightarrow \sim p$

∴ It is false that if it will rain tomorrow, then it will rain today.

(iv) let : p = Ubaya committed crime

q = He will be at the crime scene

Statement : $p \rightarrow q$

Contrapositive : $\sim q \rightarrow \sim p$

∴ It is false that if Ubaya will be at the crime scene, then he committed a crime.

b) $(P \wedge \sim Q) \vee ((\sim P \vee R) \wedge Q)$ .... given proposition.

$\equiv (P \wedge \sim Q) \vee (\sim P \wedge Q) \vee (R \wedge Q)$ .... Distributive law.

$\equiv [(P \wedge \sim Q) \vee (\sim P \wedge Q)] \vee (R \wedge Q)$ ... Associative law.

$\equiv [(P \wedge \sim Q) \vee \sim P] \wedge [(P \wedge \sim Q) \vee Q] \vee (R \wedge Q)$ .... Distributive law.

$\equiv [(\sim P \vee P) \wedge (\sim P \vee Q)] \wedge [(P \vee Q) \wedge (Q \vee \sim Q)] \vee (R \wedge Q)$ ... Distributive law

$\equiv (T \wedge (\sim P \vee Q)) \wedge ((P \vee Q) \wedge T) \vee (R \wedge Q)$ ... complement law.

$\equiv [(\sim P \vee Q) \wedge (P \vee Q)] \vee (R \wedge Q)$ .. identity law.

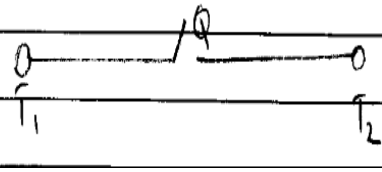
$\equiv Q \vee (\sim P \wedge P) \vee (R \wedge Q)$ .... Distributive law.

$\equiv (Q \vee F) \vee (R \wedge Q)$ ..... complement law.

$\equiv Q \vee (R \wedge Q)$ .... identity law.

$\equiv Q$ ... Absorption law.

Electrical network.



2. ©. $(\sim P \wedge \sim Q) \rightarrow (Q \rightarrow R)$

To eliminate $\rightarrow$

$\sim(\sim P \wedge \sim Q) \wedge (\sim Q \wedge R)$ ------- by implification definition.

$(P \vee Q) \wedge (\sim Q \wedge R)$ ---------- De-morgan's law.

$((P \vee Q) \wedge \sim Q) \wedge ((P \vee Q) \wedge R)$ ------ Distributive law.

$(P \wedge \sim Q) \vee (Q \wedge \sim Q) \wedge (P \wedge R) \vee (Q \wedge R)$ --- distributive law.

$(P \wedge \sim Q) \vee F \wedge (P \wedge R) \vee (Q \wedge R)$ -------- complement law.

$(P \wedge \sim Q) \wedge (P \wedge R) \vee (Q \wedge R)$ ------- Identity law.

$P \wedge (\sim Q \wedge R) \vee (Q \wedge R)$ --------- distributive law.

$P \wedge (P \sim Q \vee Q) \wedge R$ ------------ distributive law.

$P \wedge T \wedge R$ ------------------ complement law

$P \wedge R$ ------------------ Identity law.

ⓓ. let: P – I like mathematics

q – I will study hard

r – I fail.

" $p \rightarrow q$, $q \vee r$, $r \rightarrow \sim p$

$[(p \rightarrow q) \wedge (q \vee r)] \rightarrow (r \rightarrow \sim p)$ ------ given;

$[(\sim p \vee q) \wedge (q \vee r)] \rightarrow (\sim r \vee \sim p)$ ------ by definition

$\sim[(\sim p \vee q) \wedge (q \vee r)] \vee (\sim r \vee \sim p)$ ------- by definition.

$\sim(\sim p \vee q) \vee \sim(q \vee r) \vee (\sim r \vee \sim p)$ ------- Demorgan's law.

$(p \wedge \sim q) \vee (\sim q \wedge \sim r) \vee (\sim r \vee \sim p)$ ------- Demorgan's law.

$(p \wedge \sim q) \vee (\sim r \wedge \sim q) \vee (\sim r \vee \sim p)$ -------- commutative law.

$(p \vee \sim r) \wedge \sim q \vee (\sim r \vee \sim p)$ ---------- distributive law.

$(p \wedge \sim q) \vee (\sim r \wedge \sim q) \vee (\sim r \vee \sim p)$ ------- distributive law.

$(p \wedge \sim q) \vee (\sim r \vee \sim p) \vee (\sim r \wedge \sim q)$ ------- commutative law.

$((p \wedge \sim q) \vee \sim r) \vee (p \wedge \sim q) \vee \sim p \vee (\sim r \wedge \sim q)$ --- distributive law.

$(p \vee \sim r) \wedge (\sim q \vee \sim r) \vee (p \vee \sim p) \wedge (\sim q \vee \sim p) \vee (\sim r \wedge \sim q)$ -- distributive law

$(p \vee \sim r) \wedge (\sim q \vee \sim r) \vee T \wedge (\sim q \vee \sim p) \vee (\sim r \wedge \sim q)$ --- complement law.

2. (a) $(p \lor \neg r) \land T \land (\neg q \lor \neg p) \lor (\neg r \land \neg q)$ ---------- Identity law.

$(p \lor \neg r) \land (\neg q \lor \neg p) \lor (\neg r \land \neg q)$ ---------- Identity law

$(p \lor \neg r) \land \neg q \lor (p \lor \neg r) \land \neg p \lor (\neg r \land \neg q)$ ----- distributive law

$(p \land \neg q) \lor (\neg r \land \neg q) \lor (p \land \neg p) \lor (\neg r \land \neg p) \lor (\neg r \land \neg q)$ --- distributive law.

$(p \land \neg q) \lor (\neg r \land \neg q) \lor F \lor (\neg r \land \neg p) \lor (\neg r \land \neg q)$ --- complement law.

$(p \land \neg q) \lor F \lor (\neg r \land \neg p) \lor (\neg r \land \neg q) \lor (\neg r \land \neg q)$ ---- commutative law.

$(p \land \neg q) \lor F \lor (\neg r \land \neg p) \lor (\neg r \land \neg q)$ ------- Idempotent law.

$(p \land \neg q) \lor (\neg r \land \neg p) \lor (\neg r \land \neg q)$ ------ Identity law

$(p \land \neg q) \lor (\neg p \lor \neg q) \land \neg r$ ---- distributive law

$((p \land \neg q) \lor \neg p \lor (p \land \neg q) \lor \neg q) \land \neg r$ ... distributive law

$(p \lor \neg p) \land (\neg q \lor \neg p) \lor (p \land \neg q) \lor \neg q) \land \neg r$ --- distributive law.

$(T \land (\neg q \lor \neg p) \lor (p \land \neg q) \lor \neg q) \land \neg r$ --- complement law.

$[T \land (\neg q \lor \neg p) \lor \neg q] \land \neg r$ ----- absorption law

$(\neg q \lor \neg p) \lor \neg q) \land \neg r$ ------.

Extract 12.2: A sample of incorrect responses to question 2 of paper 2

In Extract 12.2, part (a), the candidate wrote incorrect contrapositive statements. In part (b), the candidate unnecessarily and incorrectly simplified the given proposition, leading to an incorrect diagram of the electrical network. In part (c), he/she failed to use the algebraic properties of sets to eliminate "$\rightarrow$" from the statement $(\sim P \land \sim Q) \rightarrow (Q \rightarrow R)$. In part (d), the candidate failed to test the validity of the given argument.

2.2.3 Question 3: Vectors

This question examined the candidates' competence in the concepts of derivatives and integration of vectors, as well as the application of the cross product to find the area of a parallelogram. The question had three parts: (a), (b), and (c), which where stated as follows:

(a) *Suppose* $\underset{\sim}{p} = t\underset{\sim}{i} + t^2\underset{\sim}{j} + 2t\underset{\sim}{k}$ *and* $\underset{\sim}{q} = (1+t^2)i + (2-t)\underset{\sim}{j} + 3\underset{\sim}{k}$, *find the derivative of* $\underset{\sim}{p} \bullet \underset{\sim}{q}$ *with respect to t.*

(b) *The velocity of a body at time t is given by* $\underset{\sim}{v} = 3t^2\underset{\sim}{i} - 2t\underset{\sim}{j} + 4\underset{\sim}{k}$. *Find the expression for the acceleration* $\underset{\sim}{a}$ *and the displacement* $\underset{\sim}{s}$ *of the body at time t, given that* $\underset{\sim}{s} = 3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ *when* $t = 1$.

(c) *Determine the area of the parallelogram ABCD with vertices* $A(1,-2,3)$, $B(4,3,1)$, $C(2,2,1)$ *and* $D(5,7,-3)$.

The data analysis shows that this question was attempted by 18,134 (100%) candidates. Among them, 5,899 (32.53%) scored 0 to 5.0 marks, 5,485 (30.25%) scored 5.5 to 8.5 marks, and 6,750 (37.22%) scored 9.0 to 15.0 marks. Figure 14 shows a summary of the candidates' performance on this question.

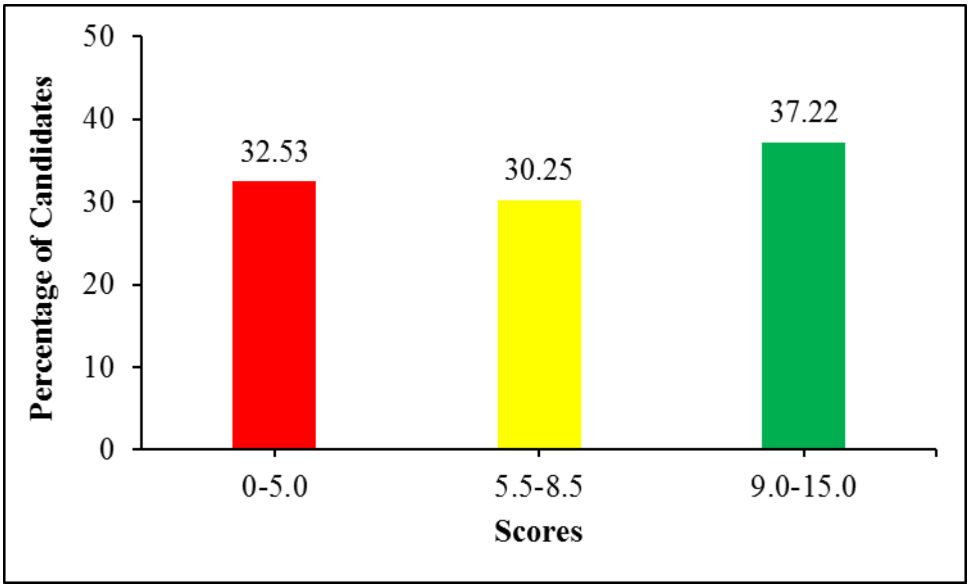


Figure 14: *The Candidates' Performance in Question 3 of Paper 2*

As Figure 14 shows, 12,235 (67.47%) candidates scored 5.5 marks and above, indicating a good performance on this question. Analysis of the responses shows that the candidates who performed well on this question demonstrated good competence in the concept of vector derivatives and the application of the cross product in finding the area of a parallelogram. In part (a), the candidates found the derivative of the dot product of the vectors $\underset{\sim}{p} = t\underset{\sim}{i} + t^2\underset{\sim}{j} + 2t\underset{\sim}{k}$ and $\underset{\sim}{q} = (1+t^2)i + (2-t)\underset{\sim}{j} + 3\underset{\sim}{k}$. First, they computed $\underset{\sim}{p} \bullet \underset{\sim}{q}$ to get $\underset{\sim}{p} \bullet \underset{\sim}{q} = 7t + 2t^2$ and then differentiated it to get

$\frac{d}{dt}\left(\underset{\sim}{p} \bullet \underset{\sim}{q}\right) = 7 + 4t$. In part (b), the candidates found the acceleration and displacement from the given velocity vector $\underset{\sim}{v} = 3t^2\underset{\sim}{i} - 2t\underset{\sim}{j} + 4\underset{\sim}{k}$. To find the acceleration, the candidates differentiated the velocity vector by writing $\underset{\sim}{a} = \frac{d}{dt}\left(3t^2\underset{\sim}{i} - 2t\underset{\sim}{j} + 4\underset{\sim}{k}\right)$ to get the acceleration vector $\underset{\sim}{a} = 6t\underset{\sim}{i} - 2\underset{\sim}{j}$, and upon substituting $t = 1$, they obtained $\underset{\sim}{a} = 6\underset{\sim}{i} - 2\underset{\sim}{j}$. To find the displacement, the candidates integrated the velocity vector by writing $\underset{\sim}{s} = \int\left(3t^2\underset{\sim}{i} - 2t\underset{\sim}{j} + 4\underset{\sim}{k}\right)dt$ to get the displacement vector $\underset{\sim}{s} = t^3\underset{\sim}{i} - t^2\underset{\sim}{j} + 4t\underset{\sim}{k} + \underset{\sim}{c}$. Finally, they obtained $\underset{\sim}{s} = \left(t^3 + 2\right)\underset{\sim}{i} - t^2\underset{\sim}{j} + \left(4t - 2\right)\underset{\sim}{k}$ after substituting $t = 1$ and $\underset{\sim}{s} = 3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$. In part (c), the candidates determined the area of the parallelogram by using the formula $A = \left|\overrightarrow{AD} \times \overrightarrow{AB}\right|$, where $\overrightarrow{AD}$ and $\overrightarrow{AB}$ are vectors of any two adjacent sides. Others split the given parallelogram into two triangles, calculated the areas of the triangles, and summed them to obtain the required answer. Extract 13.1 is a sample of correct responses from one of the candidates who answered the question correctly.

3 a) Soln.

Given, $\underline{p} = t\underline{i} + t^2\underline{j} + 2t\underline{k}$.

$\underline{q} = (1+t^2)\underline{i} + (2-t)\underline{j} + 3\underline{k}$.

$\underline{p}\cdot\underline{q} = (t\underline{i} + t^2\underline{j} + 2t\underline{k})\cdot((1+t^2)\underline{i} + (2-t)\underline{j} + 3\underline{k})$

$\underline{p}\cdot\underline{q} = t(1+t^2) + t^2(2-t) + 2t(3)$

$\underline{p}\cdot\underline{q} = t + t^3 + 2t^2 - t^3 + 6t$

$\underline{p}\cdot\underline{q} = 2t^2 + 7t$.

$\frac{d(\underline{p}\cdot\underline{q})}{dt} = 4t + 7$.

$\therefore \frac{d(\underline{p}\cdot\underline{q})}{dt} = 4t + 7$.

b) Soln.

given, $\underline{V}(t) = 3t^2\underline{i} - 2t\underline{j} + 4\underline{k}$.

for acceleration,

$a_{(t)} = \frac{dV_{(t)}}{dt} = 6t\underline{i} - 2\underline{j}$

3 b/ Soln.

For displacement,

$$\underline{s}_{(t)} = \int V_{(t)}\,dt = \int (3t^2\underline{i} - 2t\underline{j} + 4\underline{k})\,dt$$

$$\underline{s}_{(t)} = t^3\underline{i} - t^2\underline{j} + 4t\underline{k} + c$$

At t = 1

$$s_{(1)} = \underline{i} - \underline{j} + 4\underline{k} + c$$

$$3\underline{i} - \underline{j} + 2\underline{k} = \underline{i} - \underline{j} + 4\underline{k} + c$$

$$c = 2\underline{i} - 2\underline{k}$$

then,

$$s_{(t)} = t^3\underline{i} - t^2\underline{j} + 4t\underline{k} + 2\underline{i} - 2\underline{k}$$

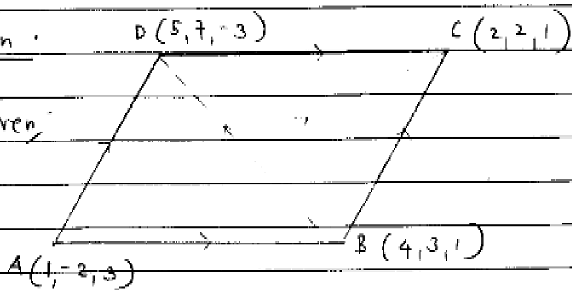
$$s_{(t)} = (t^3+2)\underline{i} - t^2\underline{j} + (4t-2)\underline{k}$$

$\therefore$ Acceleration, $\underline{a_{(t)} = 6t\underline{i} - 2\underline{j}}$

Displacement, $\underline{S_{(t)} = (t^3+2)\underline{i} - t^2\underline{j} + (4t-2)\underline{k}}$

c/ Soln.

Given,



3c) Soln

from, $A = |\underset{\sim}{a} \times \underset{\sim}{b}|$

$\underset{\sim}{a} = \overline{AB} = (4, 3, 1) - (1, -2, 3)$

$\overline{AB} = (3, 5, -2)$

$\overline{AB} = 3i + 5j - 2k$

$\underset{\sim}{b} = \overline{AD} = (5, 7, -3) - (1, -2, 3)$

$\overline{AD} = (4, 9, -6)$

$\overline{AD} = 4i + 9j - 6k$

$A = \left| (3\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k}) \times (4\underset{\sim}{i} + 9\underset{\sim}{j} - 6\underset{\sim}{k}) \right|$

From,

$$\overline{AB} \times \overline{AD} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & 5 & -2 \\ 4 & 9 & -6 \end{vmatrix}$$

$\overline{AB} \times \overline{AD} = \underset{\sim}{i}(-30 + 18) - \underset{\sim}{j}(-18 + 8) + \underset{\sim}{k}(27 - 20)$

$\overline{AB} \times \overline{AD} = -12\underset{\sim}{i} + 10\underset{\sim}{j} + 7\underset{\sim}{k}$

$|\overline{AB} \times \overline{AD}| = \sqrt{(-12)^2 + (10)^2 \times 7^2}$

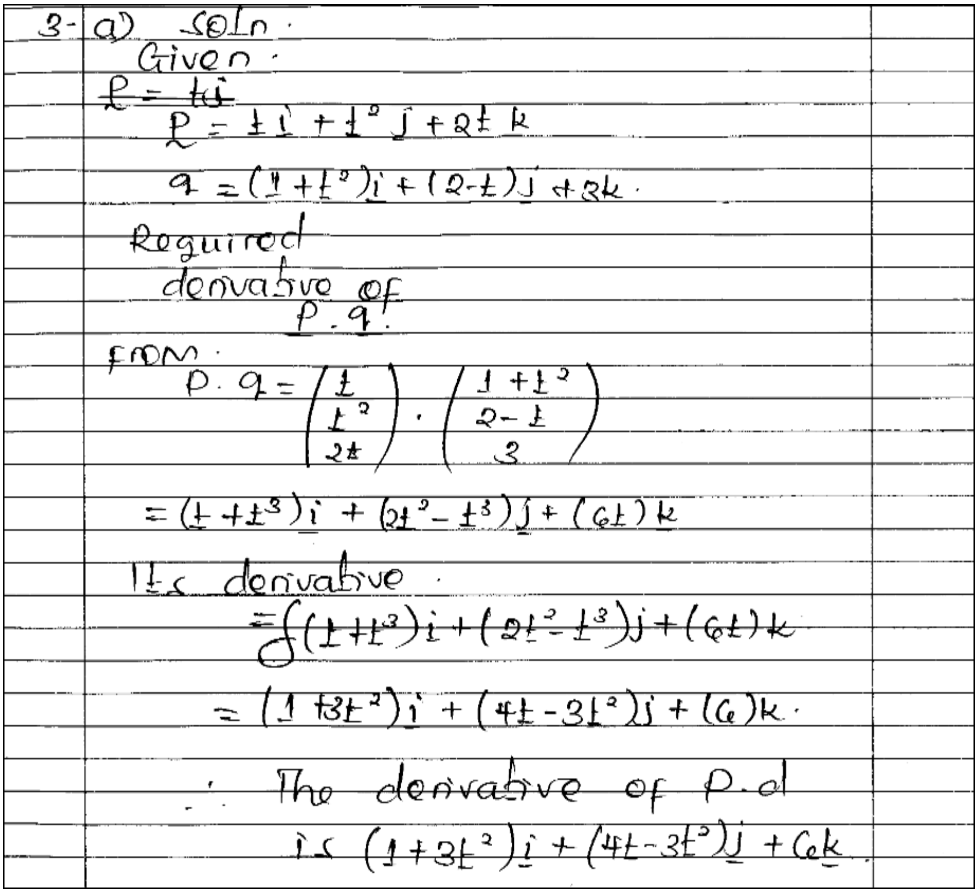
$\therefore$ Area $= \sqrt{293} = 17.12$ square units

Extract 13.1: A sample of correct responses to question 3 of paper 2

In Extract 13.1, part (a), the candidate found the dot product of the vectors and then differentiated it to obtain the correct answer. In part (b), the candidate differentiated the velocity vector and substituted $t = 1$ to get the required acceleration. He/she then integrated the velocity vector and substituted t and $\underset{\sim}{s}$ to obtain the correct displacement. In part (c), the candidate determined the area of the parallelogram by using the cross product of two adjacent sides.

Nevertheless, the data show that among the 5,899 (32.53%) candidates who scored less than 5.5 marks, 812 (4.48%) scored zero. This indicates that they lacked competence in the concept of vector derivatives and the application of the cross product in finding the area of a parallelogram. In part (a), the candidates incorrectly computed the dot product of the vectors $\underset{\sim}{p} = t\underset{\sim}{i} + t^2\underset{\sim}{j} + 2t\underset{\sim}{k}$ and $\underset{\sim}{q} = (1 + t^2)i + (2 - t)\underset{\sim}{j} + 3\underset{\sim}{k}$. For instance, some wrote

$\underline{p} \bullet \underline{q} = \begin{pmatrix} t \\ t^2 \\ 2t \end{pmatrix} \bullet \begin{pmatrix} 1+t^2 \\ 2-t \\ 3 \end{pmatrix} = (t+t^3)\underline{i} + (2t^2 - t^3)\underline{j} + 6t\underline{k}$. In part (b), the candidates applied incorrect concepts to determine acceleration and displacement from the given velocity vector. They substituted $t = 1$ into $\underset{\sim}{v} = 3t^2\underset{\sim}{i} - 2t\underset{\sim}{j} + 4\underset{\sim}{k}$ and obtained $\underline{v} = 3\underline{i} - 2\underline{j} + 4\underline{k}$. Due to this mistake, they ended up with incorrect values for both acceleration and displacement. Additionally, some candidates failed to correctly differentiate and integrate the given velocity vector. In part (c), the candidates applied an incorrect concept in determining the area of the parallelogram. For example, they used two parallel side vectors by writing $Area = \left|\overrightarrow{AB} \times \overrightarrow{CD}\right|$. Extract 13.2 is a sample of incorrect responses from one of the candidates who answered the question incorrectly.


3. a) Soln.
Given:
~~p = ti~~
$\underset{\sim}{p} = ti + t^2 j + 2t k$
$q = (1+t^2)\underline{i} + (2-t)\underline{j} + 3k$.
~~Required~~
derivative of $\underline{p}.\underline{q}$!
from:
$p.q = \begin{pmatrix} t \\ t^2 \\ 2t \end{pmatrix} . \begin{pmatrix} 1+t^2 \\ 2-t \\ 3 \end{pmatrix}$
$= (t+t^3)\underline{i} + (2t^2 - t^3)\underline{j} + (6t)k$
Its derivative:
~~$= \int (t+t^3)i + (2t^2-t^3)j + (6t)k$~~
$= (1+3t^2)\underline{i} + (4t-3t^2)\underline{j} + (6)k$.
∴ ~~The derivative of p.q~~
is $(1+3t^2)\underline{i} + (4t-3t^2)\underline{j} + 6\underline{k}$.

3 (b.) Solution

$\underset{\sim}{v} = 3t^2 \underset{\sim}{i} - 2t \underset{\sim}{j} + 4k.$

~~$\underset{\sim}{a} = \frac{d}{dt}(v)$~~

~~$\underset{\sim}{a} = \frac{d}{dt}(3t^2 \underset{\sim}{i} - 2t\underset{\sim}{j}$~~

$\int = 3\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

then.

$V = 3(3t^2)\underset{\sim}{i} - 2t(-1)\underset{\sim}{j} + 4(2)\underset{\sim}{k}$

$V = 9t^2 \underset{\sim}{i} + 2t \underset{\sim}{j} + 8\underset{\sim}{k}.$

acceleration (a) = ?

$a = \frac{d}{dt}(v)$

$a = \frac{d}{dt}(9t^2\underset{\sim}{i} + 2t\underset{\sim}{j} + 8\underset{\sim}{k})$

$a = 18t\underset{\sim}{i} + 2\underset{\sim}{j}$

acceleration expression is $18t\underset{\sim}{i} + 2\underset{\sim}{j}$

when $t = 1$,

acceleration is $18\underset{\sim}{i} + 2\underset{\sim}{j}$

3. c) Area of parallelogram.

$A = |AB \times CD|$.

Whereas

$AB = B - A$.

Whereby

$B = 4i + 3j + k$

$A = i - 2j + 3k$.

$(4i + 3j + k) - (i - 2j + 3k)$.

$3i + 5k - 2k = AB$

$CD = D - C$

whereby

$D = 5i + 7j - 3k$

$C = 2i + 2j + k$.

$(5i + 7j - 3k) - (2i + 2j + k)$.

$3i + 5j - 4k = CD$

$A = |\vec{AB} \times \vec{CD}|$.

$AB \times CD = \begin{vmatrix} i & j & k \\ 3 & 5 & -2 \\ 3 & 5 & -4 \end{vmatrix}$

$= (5 \times -4 - 5 \times -2)i - (-12 - -6)j + (15 - 15)j$

$= (-20 + 10)i + 6j + 0$

$= -10i + 6j$.

$|\vec{AB} \times \vec{CD}| = \sqrt{(-10)^2 + 6^2}$

$= \sqrt{100 + 36}$

$= \sqrt{136}$

$= 2\sqrt{34}$.

$\therefore$ Area of parallelogram is $2\sqrt{34}$ square units.

Extract 13.2: A sample of incorrect responses to question 3 of paper 2

In Extract 13.2, the candidate computed the dot product incorrectly in part (a). In part (b), he/she applied an incorrect concept to find the acceleration and displacement from the given velocity vector. In part (c), the candidate used an incorrect concept to determine the area of the parallelogram by using two parallel side vectors.

2.2.4 Question 4: Complex Numbers

This question examined the candidates' understanding of the application of De Moivre's Theorem to determine the roots of complex numbers, as well as their grasp of the concept of the locus of complex numbers. The question was presented as follows:

(a) *Use De Moivre's theorem to find the roots of the equation* $x^6 - 2x^3 + 4 = 0$.

(b) *Find the Cartesian equation of the locus of* $Z = x + iy$ *when* $|Z - 1| = 2|Z|$.

(c) *Solve for* $Z^3 = 8i$.

The data analysis reveals that a total of 18,134 (100%) attempted this question, of whom 9,811 (54.10%) scored from 0 to 5.0 marks, 6,328 (34.90%) scored from 5.5 to 8.5 marks, and 1,995 (11.00%) candidates scored from 9.0 to 15.0 marks. Figure 15 summarises the candidates' performance on this question.

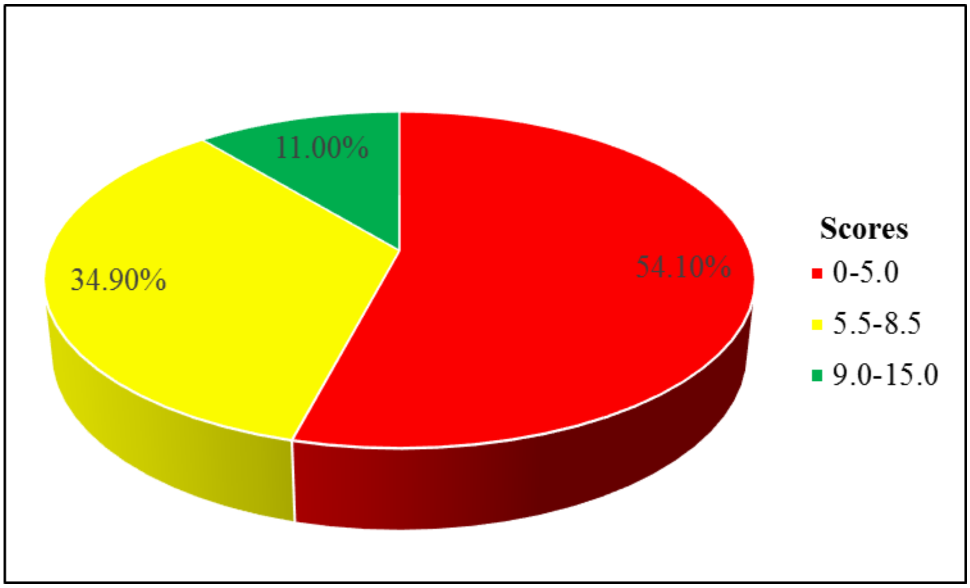


Figure 15: *The Candidates' Performance on Question 4 in Paper 2*

As Figure 15 shows, 8,323 (45.90%) candidates scored 5.5 marks or more, indicating an average performance on this question. The candidates who performed well had adequate knowledge of the application of De Moivre's Theorem to find the n^{th} roots of complex numbers, as well as to determine the locus of complex numbers. In part (a), the candidates applied De Moivre's Theorem to find the roots of the equation $x^6 - 2x^3 + 4 = 0$. First, they rewrote the equation as a quadratic equation in x^3, that is, $(x^3)^2 - 2(x^3) + 4 = 0$, and then solved it to get $x^3 = 1 \pm i\sqrt{3}$. Solving $x^3 = 1 + i\sqrt{3}$ using De Moivre's Theorem, they obtained

$x_n = \sqrt[3]{2}\left(\cos\left(\frac{\pi + 6\pi n}{9}\right) + i\sin\left(\frac{\pi + 6\pi n}{9}\right)\right)$, and hence the roots were $x_0 = \sqrt[3]{2}\left(\cos\left(\frac{\pi}{9}\right) + i\sin\left(\frac{\pi}{9}\right)\right)$, $x_1 = \sqrt[3]{2}\left(\cos\left(\frac{7\pi}{9}\right) + i\sin\left(\frac{7\pi}{9}\right)\right)$ and $\sqrt[3]{2}\left(\cos\left(\frac{13\pi}{9}\right) + i\sin\left(\frac{13\pi}{9}\right)\right)$. For $x^3 = 1 - i\sqrt{3}$, they obtained $x_n = \sqrt[3]{2}\left(\cos\left(\frac{-\pi + 6\pi n}{9}\right) + i\sin\left(\frac{-\pi + 6\pi n}{9}\right)\right)$, giving the roots $\sqrt[3]{2}\left(\cos\left(\frac{\pi}{9}\right) - i\sin\left(\frac{\pi}{9}\right)\right)$, $\sqrt[3]{2}\left(\cos\left(\frac{5\pi}{9}\right) + i\sin\left(\frac{5\pi}{9}\right)\right)$, $\sqrt[3]{2}\left(\cos\left(\frac{11\pi}{9}\right) + i\sin\left(\frac{11\pi}{9}\right)\right)$.

In part (b), the candidates found the Cartesian equation using the concept of the modulus of a complex number. From $|Z - 1| = 2|Z|$, they substituted $z = x + iy$ to obtain $|(x + iy) - 1| = 2|(x + iy)|$. Upon simplifying, they obtained $3x^2 + 3y^2 + 2x - 1 = 0$. In part (c), the candidates applied De Moivre's Theorem to solve the equation $Z^3 = 8i$. They wrote $Z_k = 2\left(\cos\left(\frac{\pi + 4\pi k}{6}\right) + i\sin\left(\frac{\pi + 4\pi k}{6}\right)\right)$ and then obtained $Z_0 = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right) = \sqrt{3} + i$, $Z_1 = 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right) = -\sqrt{3} + i$, and $Z_2 = 2\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right) = -2i$. Extract 14.1 presents a sample response from a candidate who answered the question correctly.

4 a) Soln.

given, $x^6 - 2x^3 + 4 = 0$

$(x^3)^2 - 2(x^3) + 4 = 0$

By general formula;

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x^3 = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$$

$$x^3 = \frac{2 \pm \sqrt{4 - 16}}{2}$$

$$x^3 = \frac{2 \pm \sqrt{-12}}{2}$$

$$x^3 = \frac{2 \pm 2\sqrt{3}\,i}{2}$$

$$x^3 = 1 \pm \sqrt{3}\,i$$

$x^3 = 1 + \sqrt{3}\,i$ and $x^3 = 1 - \sqrt{3}\,i$

Consider, $x^3 = 1 + \sqrt{3}\,i$

In polar form

$$Arg(\theta) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$$

$$Arg(\theta) = 60^\circ$$

$$|x^3| = \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2.$$

4-9) $x^3 = 2(\cos 60° + i\sin 60°)$

$x = [2(\cos 60° + i\sin 60°)]^{1/3}$

$x = 2^{1/3}(\cos 60° + i\sin 60°)^{1/3}$

$x_{k+1} = 2^{1/3}\left(\cos\left(\frac{60° + 360°k}{3}\right) + i\sin\left(\frac{60° + 360°k}{3}\right)\right)$

De Moivres Theorem

For $k = 0$

$x_1 = 2^{1/3}\left[\cos\left(\frac{60°}{3}\right) + i\sin\left(\frac{60°}{3}\right)\right]$

$x_1 = 2^{1/3}[\cos(20°) + i\sin(20°)]$.

for $k = 1$

$x_2 = 2^{1/3}\left[\cos\left(\frac{420°}{3}\right) + i\sin\left(\frac{420°}{3}\right)\right]$

$x_2 = 2^{1/3}[\cos(140°) + i\sin(140°)]$.

for $k = 2$

$x_3 = 2^{1/3}\left[\cos\left(\frac{780°}{3}\right) + i\sin\left(\frac{780°}{3}\right)\right]$

$x_3 = 2^{1/3}[\cos(260°) + i\sin(260°)]$.

Consider $x^3 = 1 - \sqrt{3}i$

In polar form

$Arg(\theta) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right)$.

$Arg(\theta) = -60°$

$|x^3| = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{4} = 2$

4a)

$$x^3 = 2\left[\cos(-60^\circ) + i\sin(-60^\circ)\right]^{1/3}$$

$$x = 2^{1/3}\left[\cos(-60^\circ) + i\sin(-60^\circ)\right]$$

$$x_{k+1} = 2^{1/3}\left[\cos\left(\frac{-60^\circ+360k}{3}\right) + i\sin\left(\frac{-60+360k}{3}\right)\right]$$

⟶ De Moivres theorem.

for k = 0

$$x_1 = 2^{1/3}\left[\cos\left(\frac{-60^\circ}{3}\right) + i\sin\left(\frac{-60^\circ}{3}\right)\right]$$

$$x_1 = 2^{1/3}\left[\cos(20^\circ) - i\sin(20^\circ)\right]$$

For k = 1

$$x_2 = 2^{1/3}\left[\cos\left(\frac{300^\circ}{3}\right) + i\sin\left(\frac{300^\circ}{3}\right)\right]$$

$$x_2 = 2^{1/3}\left[\cos(100^\circ) + i\sin(100^\circ)\right]$$

For k = 2

$$x_3 = 2^{1/3}\left[\cos\left(\frac{660^\circ}{3}\right) + i\sin\left(\frac{660^\circ}{3}\right)\right]$$

$$x_3 = 2^{1/3}\left[\cos(220^\circ) + i\sin(220^\circ)\right]$$

∴ The roots are;

$$x_1 = 2^{1/3}\left[\cos(20^\circ) + i\sin(20^\circ)\right]$$

$$x_2 = 2^{1/3}\left[\cos(140^\circ) + i\sin(140^\circ)\right]$$

$$x_3 = 2^{1/3}\left[\cos(260^\circ) + i\sin(260^\circ)\right]$$

$$x_4 = 2^{1/3}\left[\cos(20^\circ) - i\sin(20^\circ)\right]$$

4)

$x_5 = 2^{1/3}\left[\cos(100^\circ) + i\sin(100^\circ)\right]$

$x_6 = 2^{1/3}\left[\cos(220^\circ) + i\sin(220^\circ)\right]$.

b) Soln.

given, $z = x + iy$

$|z-1| = 2|z|$.

$|x+iy-1| = 2|x+iy|$

$|(x-1)+iy| = 2|x+iy|$

$\sqrt{(x-1)^2+y^2} = 2\sqrt{x^2+y^2}$

$(x-1)^2+y^2 = 4(x^2+y^2)$

$x^2-2x+1+y^2 = 4x^2+4y^2$

$0 = 3x^2+3y^2+2x-1$

$\therefore$ the locus of z is $3x^2+3y^2+2x-1=0$.

c) Soln.

given, $z^3 = 8i$

Modulus $|z^3| = \sqrt{8^2+0^2}$

$|z^3| = 8$.

$Arg(\theta) = \tan^{-1}\left(\frac{y}{x}\right)$

$\theta = \tan^{-1}\left(\frac{8}{0}\right) = 90^\circ$.

In polar form,

4 c)

$z^3 = 8(\cos 90^\circ + i\sin 90^\circ)$

$z = 8^{1/3}(\cos 90^\circ + i\sin 90^\circ)^{1/3}$

$z_{k+1} = 2\left(\cos\left(\frac{90+360k}{3}\right) + i\sin\left(\frac{90+360k}{3}\right)\right)$

$\rightarrow$ De Moivre's Theorem

For $k=0$

$z_1 = 2\left[\cos\left(\frac{90^\circ}{3}\right) + i\sin\left(\frac{90^\circ}{3}\right)\right]$

$z_1 = 2\left[\cos 30^\circ + i\sin 30^\circ\right]$

$z_1 = 2\left[\frac{\sqrt{3}}{2} + \frac{1}{2}i\right]$

$z_1 = \sqrt{3} + i$

For $k=1$

$z_2 = 2\left[\cos\left(\frac{450^\circ}{3}\right) + i\sin\left(\frac{450^\circ}{3}\right)\right]$

$z_2 = 2\left[\cos(150^\circ) + i\sin(150^\circ)\right]$

$z_2 = 2\left[\frac{-\sqrt{3}}{2} + \frac{1}{2}i\right]$

$z_2 = -\sqrt{3} + i$

For $k=2$

$z_3 = 2\left[\cos\left(\frac{810^\circ}{3}\right) + i\sin\left(\frac{810^\circ}{3}\right)\right]$

$z_3 = 2\left[\cos(270^\circ) + i\sin(270^\circ)\right]$

$z_3 = 2\left[0 + -1i\right]$

4 c)

$z_3 = -2i$

$\therefore$ The values of z are $z = \sqrt{3} + i$

$z = -\sqrt{3} + i$

$z = -2i$

Extract 14.1: A sample of correct responses to question 4 of paper 2

In Extract 14.1, part (a), the candidate correctly applied De Moivre's Theorem to find the roots of the equation $x^6 - 2x^3 + 4 = 0$. In part (b), the candidate correctly found the Cartesian equation using the concept of the modulus of a complex number. In part (c), the candidate correctly applied De Moivre's Theorem to solve the given equation.

On the other hand, analysis of the responses shows that the candidates who failed to answer this question correctly had the following weaknesses. In part (a), the candidates failed to apply De Moivre's Theorem correctly and also failed to express the equation $x^6 - 2x^3 + 4 = 0$ as a quadratic equation in x^3. In part (b), the candidates failed to apply the concept of the modulus of a complex number to find the Cartesian equation of $|Z-1| = 2|Z|$. They substituted $z = x + iy$ to get $|(x+iy)-1| = 2|(x+iy)|$. Then, they wrote $\frac{x-1+iy}{x+iy} = 2$ and ended up with a wrong answer, $x^2 + y^2 + x - iy = 0$. In part (c), the candidates applied De Moivre's Theorem incorrectly to solve the equation $Z^3 = 8i$. They wrote $z^3 = r^3(\cos 3\theta + i\sin 3\theta) = r^3\cos 3\theta + ir^3\sin 3\theta$ and then compared it with $r^3 = 8$ to obtain $r^3\sin 3\theta = 8$ and $r^3\cos 3\theta = 0$. From this comparison, they calculated the value of θ and obtained $\theta = 90^\circ$. Then, they used these incorrect values and ended up with incorrect answers. Extract 14.2 is a sample of an incorrect response from one of the candidates who answered the question incorrectly.

4. (a) soln

$x^6 - 2x^3 + 4 = 0$

from Demoivres theorem

$(\cos\theta n + i\sin\theta n) = (\cos\theta + i\sin\theta)^n$

$(\cos\theta + i\sin\theta)^6 = \cos^6\theta + 6\cos^5\theta i\sin\theta + 15\cos^4\theta i^2\sin^2\theta$

$+ 20\cos^3\theta i^3\sin^3\theta + 15\cos^2\theta i^4\sin^4\theta + 6\cos\theta i^5\sin^5\theta + i^6\sin^6\theta$

$= \cos^6\theta + 6\cos^5\theta i\sin\theta - 15\cos^4\theta\sin^2\theta - 20i\cos^3\theta\sin^3\theta$

$+ 15\cos^2\theta\sin^4\theta - 6i\cos\theta\sin^5\theta + \sin^6\theta$

$\cos^6\theta = \cos^6\theta + \sin^6\theta + 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta$

$+ \sin^6\theta$

$x^6 - 2x^3 + 4 = 0$

$x^3(x^3 - 2) + 4 = 0$

$x^3(x^3 - 2) = (0x^2 - 4)$

$x^3 = 0$

$x^3 - 2 = -4$

$x^3 = -2$

$x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$

$$\begin{array}{r} 1 \\ x^6 - 2x^3 + 4 \;\big)\; x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1 \\ - \; x^6 + 0x^5 + 0x^4 + 2x^3 + 0x^2 + 0x + 4 \\ \hline 6x^5 + 15x^4 - 18x^3 + 15x^2 + 16x - 3 \end{array}$$

$\therefore$ The roots are $x^3 - 1$ and $(x^3 - 2)$ and

4 b/

$|z-1| = 2|z|$

recall

$z = x + iy$

$|x + iy - 1| = 2|x + iy|$

$|x - 1 + iy| = 2|x + iy|$

$$\frac{x-1+iy}{x+iy} = 2$$

$$\frac{x-1+iy}{x+iy} \times \frac{x-iy}{x-iy} = 2$$

$$\frac{x(x-1+iy) - iy(x-1+iy)}{x^2+y^2} = 2$$

$x^2 - x + xyi - xyi + iy + y^2 = 2x^2 + y^2$

$x^2 + y^2 - x + iy = 2x^2 + y^2$

$x^2 + y^2 + x - iy = 0$

∴ The cartesian equation of the locus is $x^2 + y^2 + x - iy = 0$

4 c) given that

$z^3 = 8i$

$z = x + iy$

$(x+iy)^3 = 8i$

$x^3 + 3x^2yi + (-3xy^2) - iy^3 = 8i$

$x^3 - 3xy^2 + 3x^2yi - iy^3 = 8i$

$$\begin{cases} x^3 - 3xy^2 = 0 \\ 3x^2y - y^3 = 8 \end{cases}$$

$x(x^2 - 3y^2) = 0$

$y(3x^2 - y^2) = 8$

$z^3 = 8i$

let

$z = r(\cos\theta + i\sin\theta)$

$z^n = r^n(\cos n\theta + i\sin n\theta)$

$z^3 = r^3(\cos 3\theta + i\sin 3\theta)$

$z^3 = 8i$

$8i = r^3\cos 3\theta + r^3 i\sin 3\theta$

$r^3\sin 3\theta = 8$ --- (i)

$r^3\cos 3\theta = 0$ --- (ii)

divide equation (i) from (ii)

$$\frac{r^3\sin 3\theta}{r^3\cos 3\theta} = \frac{8}{0}$$

$\tan 3\theta = \infty$

$3\theta = \tan^{-1}(\infty)$

$3\theta = 90°$

$\theta = 30° = \pi/6$

4 (c) for r: square equation (i) and (ii)

~~$r^3 \sin 3\theta = 8$~~

$$r^6 \sin^2 3\theta = 64$$

$$r^6 \cos^2 3\theta = 0$$

$$r^6 (\sin^2 3\theta + \cos^2 3\theta) = 64 + 0$$

$$r^6 = 64$$

$$r = \sqrt[6]{64}$$

$$r = 2$$

$$z^3 = 2(\cos 3\theta + i\sin 3\theta)$$

recall.

$$z_k = r^{1/n}\left(\cos\left(\frac{2\pi n + k\theta}{2n}\right) + \sin\left(\frac{2\pi n + k\theta}{n}\right)\right)$$

~~$k = 0, 1, 2, an$~~

$k = 0, 1$ and 2.

$$z_0 = 2^{1/3}\left(\cos\left(\frac{6\pi + \pi/3}{2}\right) + \sin\left(\frac{6\pi + \pi/3}{2}\right)\right)$$

$$z_1 = 2^{1/3}\left(\cos\left(\frac{19}{6}\pi\right) + i\sin\left(\frac{19}{6}\pi\right)\right)$$

$$z_2 = 2^{1/3}\left(\cos\left(\frac{20}{6}\pi\right) + i\sin\left(\frac{20}{6}\pi\right)\right)$$

$$z_0 = 2^{1/3}\left(\cos\left(\frac{6\pi}{3}\right) + i\sin\left(\frac{6\pi}{3}\right)\right)$$

$\therefore z_0 = 2^{1/3}\left(\cos\left(\frac{6\pi}{3}\right) + i\sin\left(\frac{6\pi}{3}\right)\right)$, $z_1 = 2^{1/3}\left(\cos\left(\frac{19}{6}\pi\right) + i\sin\left(\frac{19}{6}\pi\right)\right)$

$z_2 = 2^{1/3}\left(\cos\left(\frac{20}{6}\pi\right) + i\sin\left(\frac{20}{6}\pi\right)\right)$

Extract 14.2: A sample of incorrect responses to question 4 of paper 2

In Extract 14.2, part (a), the candidate failed to express the given equation in quadratic form in x^3 and incorrectly applied De Moivre's Theorem to solve the equation. In part (b), the candidate failed to apply the concept of modulus to find the Cartesian equation. In part (c), the candidate again applied De Moivre's Theorem incorrectly to solve the equation $Z^3 = 8i$.

2.2.5 Question 5: Trigonometry

The question tested the candidates' competence in solving trigonometric equations, expanding trigonometric functions of multiple angles, and proving trigonometric identities. The question stated that:

(a) *Solve the equation* $3+2\sin 2\theta = 2\sin\theta + 3\cos^2\theta$ *for* $0° \le \theta \le 180°$.

(b) *Find the expression of* $\sin 3x$ *in terms of* $\sin x$.

(c) *Prove that* $\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$.

(d) *Solve the equation* $5\cos x - 2\sin x = 2$ *for* $-360^\circ \le \theta \le 360^\circ$ *using the substitution* $t=\tan\left(\frac{x}{2}\right)$.

A total of 15,881 (100%) candidates responded to this question. Among them, 3,836 (24.15%) scored 0 to 6.5 marks, while 12,045 (75.85%) scored 7.0 to 20 marks. Therefore, the overall performance of candidates on this question was good. Figure 16 shows the percentage of candidates with weak, average, and good performance levels.

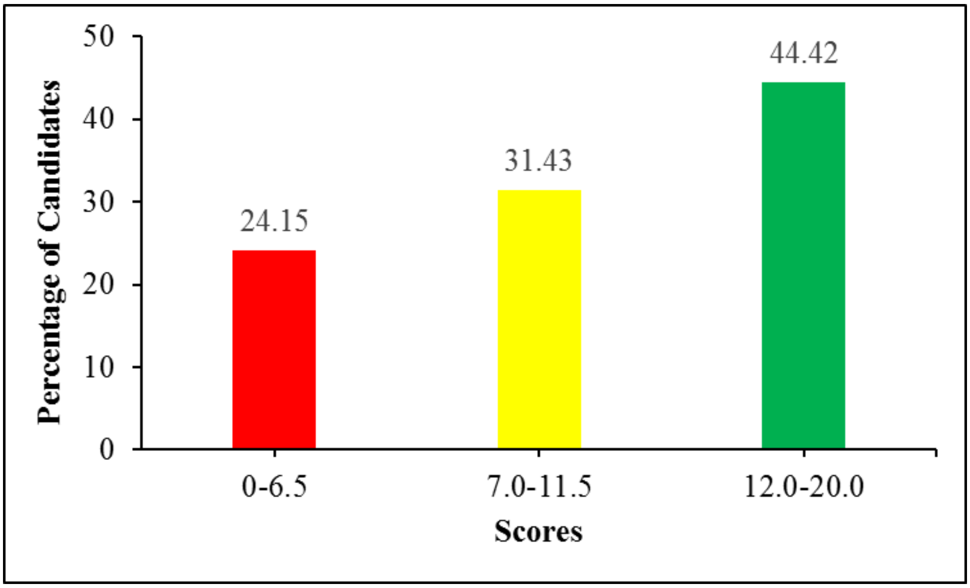


Figure 16: *The Candidates' Performance in Question 5 of Paper 2*

Analysis of the responses shows that the candidates who performed well on this question applied appropriate concepts in solving trigonometric equations. In part (a), the candidates solved the equation

$3+2\sin 2\theta = 2\sin\theta + 3\cos^2\theta$ by substituting the identities $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos^2\theta = 1-\sin^2\theta$. After substituting and simplifying, they obtained $\sin\theta(3\sin\theta + 4\cos\theta - 2) = 0$. They factorised this to get $\sin\theta = 0$ and $3\sin\theta + 4\cos\theta - 2 = 0$. Upon solving the two equations, they ended up with the correct answers of $0^\circ, 29.6^\circ, 103.29^\circ$, and 180°. In part (b), the candidates expanded $\sin 3x$ using compound angle and double angle formulae. They wrote $\sin 3x = \sin(2x + x)$ and expanded it to obtain $\sin 3x = 3\sin x - 4\sin^3 x$.

In part (c), the candidates were able to prove the identity $\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$. They let: $\alpha = \tan^{-1}\left(\frac{1}{5}\right)$ and $\beta = \tan^{-1}\left(\frac{1}{7}\right)$, $\lambda = \tan^{-1}\left(\frac{1}{3}\right)$ and $\omega = \tan^{-1}\left(\frac{1}{8}\right)$, resulted to $\tan\alpha = \frac{1}{5}$, $\tan\beta = \frac{1}{7}$ $\tan\lambda = \frac{1}{3}$, and $\tan\omega = \frac{1}{8}$ respectively. Then, they substituted the variables to get $\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \alpha + \beta + \lambda + \omega$, then $\alpha + \beta + \lambda + \omega = \tan^{-1}\tan(\alpha + \beta + \lambda + \omega)$. On further simplification, they obtained the required identity $\alpha + \beta + \lambda + \omega = \tan^{-1}(1) = \frac{\pi}{4}$. Hence, proved that $\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$.

In part (d), the candidates solved the equation $5\cos x - 2\sin x = 2$ by using the t-formulae. They recalled that $\sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$, where $t = \tan\left(\frac{x}{2}\right)$. Upon substitution, they obtained $5\left(\frac{1-t^2}{1+t^2}\right) - 2\left(\frac{2t}{1+t^2}\right) = 2$ which is equivalent to $7t^2 + 4t - 3 = 0$. After solving the equation $7t^2 + 4t - 3 = 0$, they obtained $t = \frac{3}{7}$ or -1, thus $\tan\left(\frac{x}{2}\right) = \frac{3}{7}$ or $\tan\left(\frac{x}{2}\right) = -1$. Thereafter they solved the equations $\tan\left(\frac{x}{2}\right) = \frac{3}{7}$ and

$\tan\left(\frac{x}{2}\right) = -1$ to obtain the correct answers $x = -313.60^\circ$, $46.40^\circ, -90^\circ$, and 270°. Extract 15.1 shows a sample of a correct response from one of the candidates who answered this question correctly.

5 a) Soln.

given, $3 + 2\sin 2\theta = 2\sin\theta + 3\cos^2\theta$.

$3 + 4\sin\theta\cos\theta = 2\sin\theta + 3\cos^2\theta$.

$3 - 3\cos^2\theta = 2\sin\theta - 4\sin\theta\cos\theta$.

$3(1 - \cos^2\theta) = 2\sin\theta - 4\sin\theta\cos\theta$

$3\sin^2\theta = 2\sin\theta - 4\sin\theta\cos\theta$.

$3\sin^2\theta + 4\sin\theta\cos\theta - 2\sin\theta = 0$

$\sin\theta(3\sin\theta + 4\cos\theta - 2) = 0$.

$\sin\theta = 0$ $\quad$ $3\sin\theta + 4\cos\theta - 2 = 0$.

Consider,

$\sin\theta = 0$.

$\theta = \sin^{-1}(0)$

$\theta = 0^\circ$

From, general formulae.

$\theta = \pi n + (-1)^n\alpha$

$\theta = \pi n + (-1)^n 0^\circ$

$\theta = 180^\circ n$

When, $n = \pm 1, 0, \ldots$

$\theta = 0^\circ, 180^\circ$

5 a) Consider, $3\sin\theta + 4\cos\theta - 2 = 0$

$$3\sin\theta = 2 - 4\cos\theta$$
$$(3\sin\theta)^2 = (2 - 4\cos\theta)^2$$
$$9\sin^2\theta = 4 - 16\cos\theta + 16\cos^2\theta$$
$$9(1-\cos^2\theta) = 4 - 16\cos\theta + 16\cos^2\theta$$
$$9 - 9\cos^2\theta = 4 - 16\cos\theta + 16\cos^2\theta$$
$$0 = 25\cos^2\theta - 16\cos\theta - 5$$

$\cos\theta = 0.86990908$ $\quad\quad$ $\cos\theta = -0.2299090834$

$\theta = \cos^{-1}(0.8699)$ $\quad\quad$ $\theta = \cos^{-1}(-0.2299)$

$\theta = 29.55^\circ$ $\quad\quad$ $\theta = 103.29^\circ$

from general formulae.

$\theta = 360^\circ n \pm \alpha$

$\theta = 360^\circ n \pm 29.55^\circ$ $\quad\quad$ $\theta = 360^\circ n \pm 103.29^\circ$

when $n = 0, \pm 1, \pm 2,$

$\theta = 29.55^\circ, 330.45^\circ$ $\quad\quad$ $\theta = 256.71^\circ, 103.29^\circ$

$\therefore$ Angles, θ are $0^\circ, 180^\circ, 103.29^\circ, 150.45$

b) Soln.

From, $\sin 3x = \sin(2x + x)$

given that

$$\sin(A+B) = \sin A\cos B + \sin B\cos A.$$

$$\sin 3x = \sin(2x)\cos(x) + \sin x\cos(2x).$$

but

$$\sin 2x = 2\cos x\sin x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$

5 b) $\sin 3x = 2\sin(x)\cos(x)\cos(x) + \sin x\left(\cos^2(x) - \sin^2(x)\right)$

$\sin 3x = 2\sin(x)\cos^2(x) + \cos^2(x)\sin(x) - \sin^3(x)$.

$\sin 3x = 3\sin(x)\cos^2(x) - \sin^3(x)$.

$\sin 3x = 3\sin(x)\left[1 - \sin^2(x)\right] - \sin^3(x)$

$\sin 3x = 3\sin(x) - 3\sin^3(x) - \sin^3(x)$

$\sin 3x = 3\sin(x) - 4\sin^3(x)$.

$\therefore \sin(3x) = 3\sin(x) - 4\sin^3(x)$.

c) Solve.

let $\tan^{-1}\left(\frac{1}{5}\right) = A$ $\quad \tan^{-1}\left(\frac{1}{7}\right) = B$

$\frac{1}{5} = \tan A$ — (i) $\quad \frac{1}{7} = \tan B$ — (ii)

$\tan^{-1}\left(\frac{1}{3}\right) = B$ $\quad \tan^{-1}(1/8) = C$

$\frac{1}{3} = \tan B$ — (ii) $\quad \frac{1}{8} = \tan C$ — (iv)

$A + B + C + D = E$

$\tan\left[(A+B) + (C+D)\right] = \tan(E)$

$$\frac{\tan(A+B) + \tan(C+D)}{1 - \tan(A+B)\tan(C+D)} = \tan E$$

then, $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

5c) $\tan(A+B) = \dfrac{\frac{1}{5}+\frac{1}{7}}{1-\left(\frac{1}{5}\times\frac{1}{7}\right)} = \dfrac{\frac{12}{35}}{\frac{34}{35}} = \dfrac{6}{17}$.

Also,

$\tan(C+D) = \dfrac{\tan C + \tan D}{1 - \tan C \tan D}$

$\tan(C+D) = \dfrac{\frac{1}{8}+\frac{1}{3}}{1-\left(\frac{1}{8}\times\frac{1}{3}\right)} = \dfrac{\frac{11}{24}}{\frac{23}{24}} = \dfrac{11}{23}$

then,

$\dfrac{\frac{6}{17}+\frac{11}{23}}{1-\left(\frac{6}{17}\times\frac{11}{23}\right)} = \tan E$

$\dfrac{\frac{325}{391}}{1-\frac{66}{391}} = \tan E$.

$\dfrac{\left(\frac{325}{391}\right)}{\left(\frac{325}{391}\right)} = \tan E$

$1 = \tan E$.

$E = \tan^{-1}(1)$

$E = \pi/4$.

5c) $\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right) = E$

$\therefore \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$

hence proved.

d) Soln:

$5\cos x - 2\sin x = 2$.

For t - formula;

$\cos x = \frac{1-t^2}{1+t^2}$

$\sin x = \frac{2t}{1+t^2}$ where $t = \tan\left(\frac{x}{2}\right)$

$\frac{5(1-t^2)}{1+t^2} - \frac{4t}{1+t^2} = 2$.

$5 - 5t^2 - 4t = 2 + 2t^2$

$0 = 7t^2 + 4t - 3$

$t = \frac{3}{7}$ $\quad t = -1$

$\tan\left(\frac{x}{2}\right) = \frac{3}{7}$ $\quad \tan\left(\frac{x}{2}\right) = -1$

$\frac{x}{2} = \tan^{-1}\left(\frac{3}{7}\right)$ $\quad \frac{x}{2} = \tan^{-1}(-1)$

$\frac{x}{2} = 23.2$ $\quad \frac{x}{2} = -45^\circ$

5d) From, general formula:

$\theta = 180^\circ n + \alpha$

$\frac{x}{2} = 180^\circ n + 23.2^\circ$ $\quad \frac{x}{2} = 180^\circ n - 45^\circ$

$x = 360^\circ n + 46.4^\circ$ $\quad x = 360^\circ n - 90^\circ$.

when $n = 0, \pm 1, \pm 2, \ldots$

$x = 46.4^\circ, -313.6^\circ$ $\quad x = 270^\circ, -90^\circ,$

$\therefore$ the values of $x = 46.4^\circ, -313.6^\circ, -90^\circ, 270^\circ$.

Extract 15.1: A sample of correct responses to question 5 of paper 2

In Extract 15.1, the candidate correctly solved the given trigonometric equations. He/she substituted suitable identities, expanded trigonometric functions of multiple angles, and correctly proved the given trigonometric identities.

Despite the good performance of most candidates, 395 (2.49%) scored zero. Analysis of the responses indicates that the candidates who performed poorly failed to apply the correct concepts in solving the question. In part (a), the candidates performed wrong computations to solve the given trigonometric equation. For instance, they transformed the equation $3+2\sin 2\theta = 2\sin\theta + 3\cos^2\theta$ into $3+4\sqrt{1-\cos^2\theta}\cos\theta = 2\sqrt{1-\cos^2\theta}+3\cos^2\theta$. Then, they incorrectly squared the equation on both sides to get a wrong equation, $9+16(1-\cos^2)\cos^2\theta = 4(1-\cos^2\theta)+9\cos^4\theta$. With this mistake, they ended up getting wrong values of $\theta = 0^\circ, 103.29^\circ, 258.46^\circ$, and 360°. Others used the incorrect identities; for instance, they used $\cos^2\theta - \sin^2\theta = 1$ instead of $\cos^2\theta + \sin^2\theta = 1$, which resulted in an incorrect answer. In part (b), some candidates used the incorrect compound angle formulae when expanding $\sin 3x$. For instance, they used $\sin(2x+x) = \sin 2x\cos x - \cos 2x\sin x$ instead of $\sin(2x+x) = \sin 2x\cos x + \cos 2x\sin x$ resulting in an incorrect answer.

In part (c), the candidates failed to write the correct expansion of the expression $\tan(A+B+C+D)$. They wrote $\tan(A+B+C+D) = \dfrac{\tan A + \tan B + \tan C + \tan D}{1-\tan A\tan B\tan C\tan D}$, which led to an incorrect proof. Furthermore, some applied the wrong compound angle formula for $\tan(\alpha+\beta)$; for example, they wrote $\tan(\alpha+\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$ instead of $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$. In part (d), some candidates recalled and used the correct t-formulae for $\sin x = \dfrac{2t}{1+t^2}$ and $\cos x = \dfrac{1-t^2}{1+t^2}$ but made computational errors that resulted in incorrect answers. Extract 15.2 shows a sample of incorrect responses from a candidate who answered the question incorrectly.

5. (a) $3 + 2\sin 2\theta = 2\sin\theta + 3\cos^2\theta$

$0 \leq \theta \leq 180°$

$3 + 2(2\sin\theta\cos\theta) = 2\sin\theta + 3\cos^2\theta.$

$3 + 4\sin\theta\cos\theta = 2\sin\theta + 3\cos^2\theta.$ --- (i)

but

$\sin^2\theta + \cos^2\theta = 1.$

$\sin^2\theta = 1 - \cos^2\theta.$

$\sin\theta = \sqrt{1-\cos^2\theta}$ ------ (ii)

5 substitute eqn (ii) into eqn (i)

$3 + 4\sqrt{(1-\cos^2\theta)}\cos\theta = 2\sqrt{1-\cos^2\theta} + 3\cos^2\theta$

then

Square both side

$3^2 + (4\sqrt{1-\cos^2\theta}\cos\theta)^2 = (2\sqrt{1-\cos^2\theta})^2 + (3\cos^2\theta)^2$

$9 + 16(1-\cos^2\theta)\cos^2\theta = 4(1-\cos^2\theta) + 9\cos^4\theta$

$9 + 16(\cos^2\theta - \cos^4\theta) = 4 - 4\cos^2\theta + 9\cos^4\theta$

$9 + 16\cos^2\theta - 16\cos^4\theta = 4 - 4\cos^2\theta + 9\cos^4\theta$

$-16\cos^4\theta - 9\cos^4\theta + 16\cos^2\theta + 4\cos^2\theta + 9 - 4$

$-25\cos^4\theta + 20\cos^2\theta + 5.$

Solve quadratic eqn.

$\cos\theta = 1.$ $\cos\theta = -0.2.$

Since negative value are ~~not~~ included

$\cos\theta = 1.$

$\theta = \cos^{-1}(1)$

$\theta = 0.$

for $\cos\theta = 1$

$\theta = 2\pi n \pm \alpha.$

when $n = 0$ $\theta = 0$

$n = 1$ $\theta = 360$

for $\cos\theta = -0.2$

$\theta = \cos^{-1}(0.2)$

$\theta = 101.54°$

for $2\pi n \pm \alpha$

when $n = 0$ $\theta = 101.54°, -101.54°$

$n = 1$ $\theta = 258.46°$

$n = 2$ $\theta = 618.46$

5	for $0 \leq \theta \leq 180$	
	The Value of θ are $0, 360°, 101.54°$	
	and $258.46°$	

5	© soln	
	Consider $L.H.S$	
	$\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$	
	let	
	$A = \tan^{-1}\frac{1}{5}$	
	$\tan A = \frac{1}{5}$	
	let	
	$B = \tan^{-1}\frac{1}{7}$	
	$\tan B = \frac{1}{7}$	
	let	
	$C = \tan^{-1}\frac{1}{3}$	
	$\tan C = \frac{1}{3}$	
	let	
	$D = \tan^{-1}\frac{1}{8}$	
	$\tan D = \frac{1}{8}$	

5(c)

$$A + B + C + D = \frac{\pi}{4}$$

Apply tan

$\tan(A+B+C+D)$

$$= \frac{\tan A + \tan B + \tan C + \tan D}{1 - \tan A \tan B \tan C \tan D}$$

$$= \frac{1/5 + 1/7 + 1/3 + 1/8}{1 - (1/5 \times 1/7 \times 1/3 \times 1/8)}$$

$$= \frac{613}{840} \div \frac{839}{840}$$

$$= \frac{613}{840} \times \frac{840}{839} = \frac{\pi}{4}$$

$$= \frac{613}{839} = \frac{\pi}{4}$$

$$\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$$

hence show

Extract 15.2: A sample of incorrect responses to question 5 of paper 2

In Extract 15.2, the candidate failed to solve the trigonometric equation due to incorrect mathematical computations and failure to express $\tan(A+B+C+D)$ in terms of $\tan A$, $\tan B$, $\tan C$ and $\tan D$.

2.2.6 Question 6: Algebra

The question examined the candidates' ability to apply the laws of logarithms in solving equations, prove mathematical statements using the principles of mathematical induction, use the concept of matrices to solve simultaneous equations, and determine the r^{th} term of a binomial expansion to find the required terms. The question was stated as follows:

(a) *Show that* $\log_{16} xy = \frac{1}{2}\log_4 x + \frac{1}{2}\log_4 y$*; hence, use it to solve simultaneously the system of the following equations* $\begin{cases} \log_{16}(xy) = 3.5 \\ \dfrac{\log_4 x}{\log_4 y} = -8 \end{cases}$

(b) *Use the principles of mathematical induction to prove that* $\sum_{r=1}^{n} r^3 = \left[\frac{1}{2}n(n+1)\right]^2$.

(c) *Use Cramer's rule to solve* $\begin{cases} 7x + y + z = -1 \\ x - 3y + 2z = 0 \\ x + 4y - 3z = 4 \end{cases}$

(d) *In the expansion of* $\left(x^2 + \frac{1}{x}\right)^n$*, the coefficients of the fourth and ninth terms are equal. Find the value of* n *and the sixth term of the expansion.*

Data analysis shows that a total of 13,910 (100%) candidates responded to this question, of whom 10,505 (75.53%) scored from 7.0 to 20 marks and 3,405 (24.47%) scored from 0 to 6.5 marks. Thus, the performance of the candidates on this question was generally good. Figure 17 shows the candidates' performance on this question.

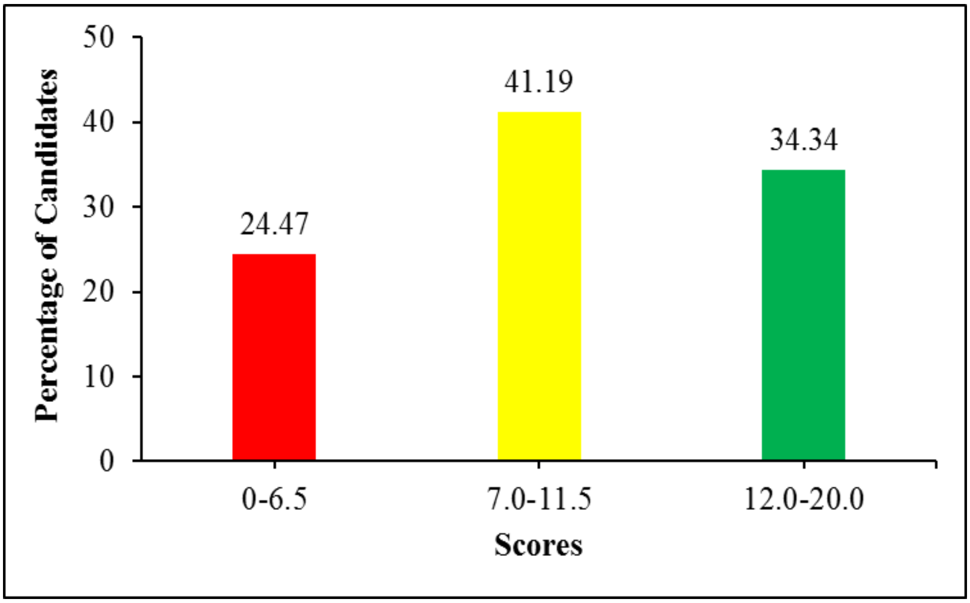


Figure 17: *The Candidates' Performance in Question 6 of Paper 2*

The response analysis shows that the candidates who performed well on this question applied appropriate algebra concepts as per the need of each item of the question. In part (a), the candidates were able to prove the identity $\log_{16} xy = \frac{1}{2}(\log_{16} x + \log_{16} y)$ by applying relevant laws of logarithms. Some candidates considered the left-hand side expression $\log_{16} xy$ and expanded it as $\log_{16} xy = \log_{16} x + \log_{16} y$. Then, they changed the base to obtain $\log_{16} xy = \frac{\log x}{\log 16} + \frac{\log y}{\log 16} = \frac{\log x}{\log 4^2} + \frac{\log y}{\log 4^2}$. On simplifying, they ended up with $\log_{16} xy = \frac{1}{2}(\log_{16} x + \log_{16} y)$ as required. Using the identity $\log_{16} xy = \frac{1}{2}(\log_{16} x + \log_{16} y)$, the candidates solved the system of simultaneous equations $\begin{cases} \log_{16}(xy) = 3.5 \\ \dfrac{\log_4 x}{\log_4 y} = -8 \end{cases}$. They re-wrote the equations as $\begin{cases} \frac{1}{2}(\log_{16} x + \log_{16} x) = \frac{7}{2} \\ \dfrac{\log_4 x}{\log_4 y} = -8 \end{cases}$, which was simplified further to

$\begin{cases} \log_4 x + \log_4 y = 7 \\ \log_4 x = -8(\log_4 y) \end{cases}$. Then, by solving simultaneously, they obtained $x = 4^8 = 65536$ and $y = \frac{1}{4}$.

In part (b), the candidates applied the principle of mathematical induction correctly to prove the statement $\sum_{r=1}^{n} r^3 = \left[\frac{1}{2}n(n+1)\right]^2$. They went through all three steps to show that the statement is true for all positive integral values of n. In step (i), they showed that the statement is true for $n = 1$. In step (ii), they assumed that the statement is true for $n = k$, such that $\sum_{1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$. In step (iii), they used the assumption to show that the statement is also true for $n = k+1$. They were able to show that $\sum_{1}^{k+1} r^3 = \left[\frac{1}{2}(k+1)(k+2)\right]^2$, which is true. This concluded the proof as required. In part (c), the candidates were able to solve the simultaneous equations by using Cramer's rule. They re-wrote the given simultaneous equations $\begin{cases} 7x + y + z = -1 \\ x - 3y + 2z = 0 \\ x + 4y - 3z = 4 \end{cases}$ in matrix form as $\begin{pmatrix} 7 & 1 & 1 \\ 1 & -3 & 2 \\ 1 & 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$.

Then, they calculated the determinants: $\det(M) = \begin{vmatrix} 7 & 1 & 1 \\ 1 & -3 & 2 \\ 1 & 4 & -3 \end{vmatrix}$,

$\det(X) = \begin{vmatrix} -1 & 1 & 1 \\ 0 & -3 & 2 \\ 4 & 4 & -3 \end{vmatrix}$, $\det(Y) = \begin{vmatrix} 7 & -1 & 1 \\ 1 & 0 & 2 \\ 1 & 4 & -3 \end{vmatrix}$ and $\det(Z) = \begin{vmatrix} 7 & 1 & -1 \\ 1 & -3 & 0 \\ 1 & 4 & 4 \end{vmatrix}$.

Using these determinants, they computed the values of x, y, and z by taking $x = \frac{\det(X)}{\det(M)}$, $y = \frac{\det(Y)}{\det(M)}$, $z = \frac{\det(Z)}{\det(M)}$, and ended up with the correct answers $(x, y, z) = (1, -3, -5)$.

In part (d), the candidates applied the formula for the general term of a binomial expansion, that is, $T_{r+1} = {}^{n}C_{r}a^{n-r}b^{r}$, to determine the fourth term, T_4, and the ninth term, T_9. They obtained $T_4 = {}^{n}C_{3}x^{2n-9}$ and $T_9 = {}^{n}C_{8}x^{2n-24}$. By equating the coefficients of T_4 and T_9, they solved for n and found $n = 11$. Using this value, they calculated the sixth term of the expansion and arrived at the correct answer, $T_6 = 462x^7$. Extract 16.2 shows a sample of a correct response from a candidate who answered this question correctly.

Qn 6(a) soln.

Given,

$= \log_{16} xy.$

$= \log_{16} x + \log_{16} y.$

$= \frac{1}{\log_x 16} + \frac{1}{\log_y 16}$

$= \frac{1}{2\log_x 4} + \frac{1}{2\log_x 4}$

$= \frac{1}{2}\left(\frac{1}{\log_x 4} + \frac{1}{\log_x 4}\right).$

$= \frac{1}{2}\left(\frac{1}{\frac{\log 4}{\log x}} + \frac{1}{\frac{\log 4}{\log y}}\right)$

$= \frac{1}{2}\left(\frac{\log x}{\log 4} + \frac{\log y}{\log 4}\right)$

$= \frac{1}{2}\left(\log_4 x + \log_4 y\right)$

$= \frac{1}{2}\log_4 x + \frac{1}{2}\log_4 y$ hence shown.

Qn 6(c)

$$\begin{cases} \log_{16} xy = 3.5 \\ \dfrac{\log_4 x}{\log_4 y} = -8 \end{cases}$$

$\log_{16} xy = \frac{1}{2}\log_4 x + \frac{1}{2}\log_4 y.$

$\frac{1}{2}\log_4 x + \frac{1}{2}\log_4 y = \frac{7}{2}$

$\log_4 x + \log_4 y = 7 \quad \ldots (i)$

$\log_4 x = -8(\log_4 y).$

$\log_4 x + 8\log_4 y = 0 \quad \ldots (ii)$

solving simultaneously.

$\log_4 x = 8 \qquad \log_4 y = -1$

$x = 4^8 \qquad y = 4^{-1}$

$x = 65536 \qquad y = \frac{1}{4}$

$\therefore$ The values of x and y

are 65536 and $\frac{1}{4}$.

Qn 6(b) <u>Step 3:</u> proving P_n true for values of $n = k+1$, considering R.H.S equal to L.H.S.

$$\sum_{r=1}^{k+1} r^3 = \left(\frac{1}{2}\left((k+1)\ (k+1+1)\right)\right)^2$$

$$= \left[\frac{1}{2}\ (k+1)(k+2)\right]^2$$

But.

$$\sum_{r=1}^{k+1} r^3 = \sum_{r=1}^{k} r^3 + (k+1)^3.$$

consider R.H.S.

$$= \sum_{r=1}^{k} r^3 + (k+1)^3.$$

$$= \left[\frac{1}{2} k(k+1)\right]^2 + (k+1)^3.$$

$$= \frac{1}{4}(k^2)(k+1)^2 + (k+1)^3.$$

$$= (k+1)^2 \left[\frac{k^2}{4} + k+1\right].$$

$$= (k+1)^2 \left[\frac{k^2+4k+4}{4}\right].$$

Qn 6(3) $= (k+1)^2 \left[\frac{(k+2)^2}{4} \right]$.

$= (k+1)^2 \left[\frac{k+2}{2} \right]^2$

$= (k+1)^2 \left[\frac{(k+1)+1}{4} \right]^2$

but $k+1 = n$

$= n^2 \left[\frac{n+1}{2} \right]^2$

$= \left[n \left(\frac{n+1}{2} \right) \right]^2$

$= \left[\frac{n}{2} (n+1) \right]^2$ hence shown!

∴ Thus P_n is true for all natural numbers n.

Qn6(i) soln.

Given.

$$\begin{cases} 7x + y + z = -1 \\ x - 3y + 2z = 0 \\ x + 4y - 3z = 4 \end{cases}$$

writing in matrix form

$$\begin{pmatrix} 7 & 1 & 1 \\ 1 & -3 & 2 \\ 1 & 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}.$$

let $A = \begin{pmatrix} 7 & 1 & 1 \\ 1 & -3 & 2 \\ 1 & 4 & -3 \end{pmatrix}$.

$$|A| = 7\begin{vmatrix} -3 & 2 \\ 4 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} + 1\begin{vmatrix} 1 & -3 \\ 1 & 4 \end{vmatrix}$$

$$|A| = 7\left((-3 \times -3) - (2 \times 4)\right) - 1\left((-3) - (2)\right) + 1\left((4) - (-3)\right)$$

$$|A| = 19.$$

$$A_x = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -3 & 2 \\ 4 & 4 & -3 \end{pmatrix}.$$

$|A_x| = ?$

Qn 6(c)

$$|A_x| = -1\begin{vmatrix} -3 & 2 \\ 4 & -3 \end{vmatrix} - 1\begin{vmatrix} 0 & 2 \\ 4 & -3 \end{vmatrix} + 1\begin{vmatrix} 0 & -3 \\ 4 & 4 \end{vmatrix}$$

$$|A_x| = 19$$

$$x = \frac{|A_x|}{|A|}$$

$$x = \frac{19}{19} = 1$$

$$A_y = \begin{pmatrix} 7 & -1 & 1 \\ 1 & 0 & 2 \\ 1 & 4 & -3 \end{pmatrix}$$

$$|A_y| = 7\begin{vmatrix} 0 & 2 \\ 4 & -3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 4 \end{vmatrix}$$

$$|A_y| = -57.$$

$$y = \frac{|A_y|}{|A|} = \frac{-57}{19} = -3$$

$$A_z = \begin{pmatrix} 7 & 1 & -1 \\ 1 & -3 & 0 \\ 1 & 4 & 4 \end{pmatrix}.$$

Qn 6(c) $|A_z| = 7\begin{vmatrix} -3 & 0 \\ 4 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 1 & 4 \end{vmatrix} + 7\begin{vmatrix} 1 & -3 \\ 1 & 4 \end{vmatrix}$

$|A_z| = -95$

$z = \frac{|A_z|}{|A|} = \frac{-95}{19} = -5$

∴ The values of x, y and z

are 1, -3 & -5

Qn 6(d) soln.

Given

$\left(x^2 + \frac{1}{x}\right)^n$.

from

$T_{r+1} = {}^nC_r (a^{n-r}) b^r$

$a = x^2 \quad b = \frac{1}{x}$

$T_4 = T_{r+1}$

$r + 1 = 4$

$r = 3$

Qn6(d)

$T_4 = {}^nC_3 \, (x^2)^{n-3} \cdot \left(\frac{1}{x}\right)^3.$

$= {}^nC_3 \, (x^{2n-6}) \left(\frac{1}{x^3}\right).$

$= {}^nC_3 \quad x^{2n-6-3}$

$T_4 = {}^nC_3 \, x^{2n-9}$

$T_9 = T_{r+1}$

$9 = r+1$

$r = 8.$

~~$T_9 = {}^nC_9 \, (x^2)^{n-8}$~~

$T_9 = {}^nC_8 \, (x^2)^{n-8} \cdot \left(\frac{1}{x}\right)^8$

$T_9 = {}^nC_8 \, (x^{2n-16}) \left(\frac{1}{x^8}\right).$

$T_9 = {}^nC_8 \, x^{2n-16-8}$

$T_9 = {}^nC_8 \quad x^{2n-24}.$

${}^nC_8 = {}^nC_3.$

$\frac{n!}{n-8} \quad \frac{n!}{(n-8)!\,8!} = \frac{n!}{(n-3)!\,3!}$

$\frac{(n-3)!}{(n-8)!} = \frac{8!}{3!}$

Qn 6(d)

$$\frac{(n-3)(n-4)(n-5)(n-6)(n-7)(n-8)!}{(n-8)!} = 6720$$

$$(n-3)(n-4)(n-5)(n-6)(n-7) = 6720$$

$$n = 11$$

$$\left(x^2 + \frac{1}{x}\right)^{11}$$

$$T_6 = T_{r+1}$$

$$r = 5.$$

$$= {}^{11}C_5 \, (x^2)^{11-5} \cdot \left(\frac{1}{x}\right)^5$$

$$= 462 \, (x^2)^6 \left(\frac{1}{x^5}\right)$$

$$= 462 \left(\frac{x^{12}}{x^5}\right)$$

$$= 462x^7.$$

$\therefore$ The sixth term is $462x^7$.

Extract 16.1: A sample of correct responses to question 6 of paper 2

In Extract 16.1, part (a), the candidate correctly applied the laws of logarithms in solving equations, while in part (b), he/she applied the principle of mathematical induction to prove the given mathematical statement. In part (c), the candidate used Cramer's Rule correctly to solve the system of simultaneous equations. In part (d), he/she applied the formula for the general term of the binomial expansion to find the required term.

However, further analysis shows that 2,103 (15.12%) candidates scored from 0 to 5.0 marks, indicating a lack of competence in the concepts of the laws of logarithms, the principle of mathematical induction, the application of matrices to solve simultaneous equations, and binomial expansion. In part (a), the candidates failed to prove the given equality because of the incorrect application of laws of logarithms. For instance, they wrote $\frac{1}{2}\log_4 x + \frac{1}{2}\log_4 y = \log_4 (xy)^{\frac{1}{2}} = \log\left(\frac{(xy)^{\frac{1}{2}}}{4}\right) = \log\left(\frac{\left((xy)^{\frac{1}{2}}\right)^2}{4^2}\right)$; thereafter they wrote $\log\left(\frac{(xy)^{\frac{1}{2}}}{4}\right) = \log\left(\frac{\left((xy)^{\frac{1}{2}}\right)^2}{4^2}\right) = \log\left(\frac{xy}{16}\right) = \log_{16} xy$. They also failed to solve the system of simultaneous equations because they could not apply appropriate laws of logarithms.

In part (b), the candidates failed to apply principles of mathematical induction correctly to prove the given mathematical statement. In step (iii) of their proof, they wrote $P(k+1) = \sum_{r=1}^{k+1} r^3$, $Q(k+1) = \frac{1}{4}(k+1)^2(k+2) = \left[\frac{1}{2}(n)(n+1)^2\right]$. In part (c), the candidates confused Cramer's Rule with the inverse matrix method when solving the given system of simultaneous equations. In part (d), most of the candidates recalled the correct formula, $T_{r+1} = {}^nC_r a^{n-r} b^r$, but made incorrect substitutions. They failed to calculate the fourth and ninth terms due to incorrect substitution of values for n. For instance, they wrote $T_4 = {}^4C_3 a^1 b^3$ and $T_9 = {}^9C_8 a^1 b^8$. Extract 16.2 shows a sample of incorrect responses from a candidate who answered this question incorrectly.

a) $\log_{16} xy = \frac{1}{2}\log_4 x + \frac{1}{2}\log_4 y$

Consider L.H.S.

$\frac{1}{2}\log_4 x + \frac{1}{2}\log_4 y = \log_4 x^{\frac{1}{2}} + \log_4 y^{\frac{1}{2}}$

$= \log_4 (x^{\frac{1}{2}} \times y^{\frac{1}{2}})$

$= \frac{\log (x^{\frac{1}{2}} \times y^{\frac{1}{2}})}{\log 4}$

6a)

$= \log \left(\frac{(xy)^{\frac{1}{2}}}{4}\right)$

By squaring both numerator and denominator

$= \log \left(\frac{[(xy)^{\frac{1}{2}}]^2}{4^2}\right)$

$= \log \left[\frac{(xy)}{16}\right]$

$= \frac{\log xy}{\log 16}$

$= \log_{16} xy$

$\therefore \log_{16} xy = \frac{1}{2}\log_4 x + \frac{1}{2}\log_4 y$

Shown!!!

$\log_{16} xy = 3.5$ ---- i)

$\frac{\log_4 x}{\log_4 y} = -8$ ---- ii)

Consider equation i)

$\log_{16} xy = 3.5 \equiv 16^{3.5} = xy$

$xy = 16^{3.5}$

$xy = 16384.$

6(a).

$$\frac{\log_4 x}{\log_4 y} = \log_y 4$$

$$= \frac{\log x}{\log 4} \div \frac{\log y}{\log 4}$$

$$= \frac{\log x}{\log 4} \times \frac{\log 4}{\log y}$$

$$= \frac{\log x}{\log y} = \log\left(\frac{x}{y}\right) = -8$$

$$\log_{10}\left(\frac{x}{y}\right) = -8$$

$$10^{-8} = \frac{x}{y}$$

$$x = 10^{-8}y$$

$$xy = 16384$$

$$x \cdot y = \frac{16384}{y}$$

$$y \times 10^{-8}y = \frac{16384}{y} \times y$$

$$\frac{10^{-8}y^2}{10^{-8}} = \frac{16384}{10^{-8}}$$

$$\sqrt{y^2} = \sqrt{1.6384 \times 10^{12.7}}$$

6(c). $7x + y + z = -1$

$x - 3y + 2z = 0$

$x + 4y - 3z = 4.$

$$\begin{pmatrix} 7 & 1 & 1 \\ 1 & -3 & 2 \\ 1 & 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}.$$

let:

$$A = \begin{pmatrix} 7 & 1 & 1 \\ 1 & -3 & 2 \\ 1 & 4 & -3 \end{pmatrix}$$

$A^{-1} = ?$

From:

$$A^{-1} = \frac{1}{|A|} \cdot \text{Adj}\, A$$

$\text{Adj}.\, A = (A^c)^T$

$$|A| = \begin{vmatrix} -3 & 2 \\ 4 & -3 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} + \begin{vmatrix} 1 & -3 \\ 1 & 4 \end{vmatrix}$$

$= (9 - 8) + (-3 - 2) + (4 - -3).$

$|A| = 19$

$$A^c = \begin{bmatrix} \begin{vmatrix} -3 & 2 \\ 4 & -3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 1 & 4 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 4 & -3 \end{vmatrix} & \begin{vmatrix} 7 & 1 \\ 1 & -3 \end{vmatrix} & \begin{vmatrix} 7 & 1 \\ 1 & 4 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ -3 & 2 \end{vmatrix} & \begin{vmatrix} 7 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 7 & 1 \\ 1 & -3 \end{vmatrix} \end{bmatrix}$$

69.

$$A^c = \begin{pmatrix} 1 & 5 & 7 \\ 7 & -22 & -27 \\ 5 & -13 & -22 \end{pmatrix}.$$

$$(A^c)^T = \begin{pmatrix} 1 & 7 & 5 \\ 5 & -22 & -13 \\ 7 & -27 & -22 \end{pmatrix}.$$

$$A^{-1} = \frac{1}{|A|} \cdot (A^c)^T.$$

$$= \frac{1}{19}\begin{pmatrix} 1 & 7 & 5 \\ 5 & -22 & -13 \\ 7 & -27 & -22 \end{pmatrix}.$$

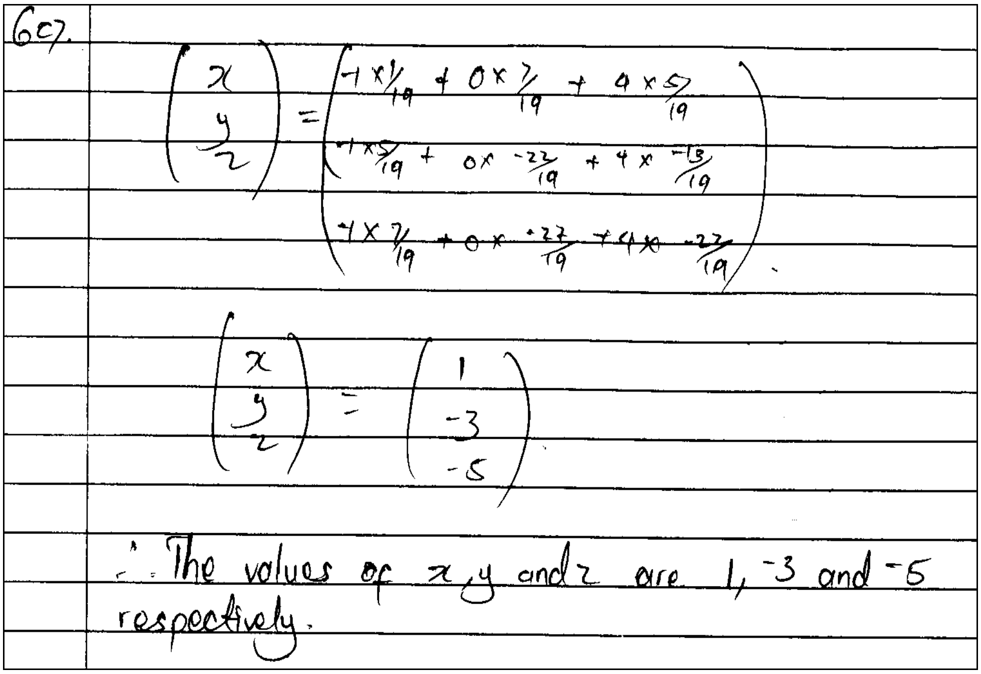
$$A^{-1} = \begin{pmatrix} 1/19 & 7/19 & 5/19 \\ 5/19 & -22/19 & -13/19 \\ 7/19 & -27/19 & -22/19 \end{pmatrix}.$$

$$\begin{pmatrix} 1/19 & 7/19 & 5/19 \\ 5/19 & -22/19 & -13/19 \\ 7/19 & -27/19 & -22/19 \end{pmatrix}\begin{pmatrix} 7 & 1 & 1 \\ 1 & -3 & 2 \\ 1 & 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}\begin{pmatrix} 1/19 & 7/19 & 5/19 \\ 5/19 & -22/19 & -13/19 \\ 7/19 & -27/19 & -22/19 \end{pmatrix}.$$

But: $(A^{-1})(A) = I$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}(A^{-1}.$$


6c).

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1\times\frac{1}{19} + 0\times\frac{7}{19} + 4\times\frac{5}{19} \\ -1\times\frac{5}{19} + 0\times\frac{-22}{19} + 4\times\frac{-13}{19} \\ -4\times\frac{7}{19} + 0\times\frac{-22}{19} + 4\times\frac{-22}{19} \end{pmatrix}.$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}$$

∴ The values of x, y and z are 1, −3 and −5 respectively.

Extract 16.2: A sample of incorrect responses to question 6 of paper 2

In Extract 16.2, part (a), the candidate failed to apply the laws of logarithms to prove the given equality and solve the simultaneous equations. In part (c), he/she used the inverse matrix method instead of Cramer's Rule to solve the simultaneous equations.

2.2.7 Question 7: Differential equations

The question examined the candidates' competence in formulating differential equations, solving first- and second-order differential equations, and applying the concepts of differential equations to solve real-life problems. The question stated that:

(a) *Find the general solution of the following differential equations:*

(i) $\frac{dy}{dx} - \frac{y}{x} = x^2$

(ii) $\frac{dy}{dx} - x\tan(y - x) = 1$

(b) *Eliminate the constants A and B in the equation* $x = A\cos(2t + B)$.

(c) *A particle P falls vertically under gravity. The air resistance of the falling particle is taken to be proportional to its speed at any instant. The distance x at which the particle has fallen after t seconds is given*

by the differential equation $\frac{d^2x}{dt^2} = g - k\frac{dx}{dt}$, *where* k *and* g *are positive constants. Find the distance* x *as a function of* t *given* $\frac{dx}{dt} = 0$ *and* $x = 0$ *when* $t = 0$.

(d) *Determine the solution of the differential equation* $(2x-1)\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$ *given that* $x = 0$, $y = 2$ *and* $\frac{dy}{dx} = 3$.

Data analysis shows that a total of 1,849 (100%) candidates responded to this question. Among them, 759 (41.05%) candidates scored from 0 to 6.5 marks, 636 (34.40%) scored from 7.0 to 11.5 marks, and 454 (24.55%) scored from 12.2 to 20.0 marks. Figure 18 presents the percentage of candidates who had weak, average, and good performances.

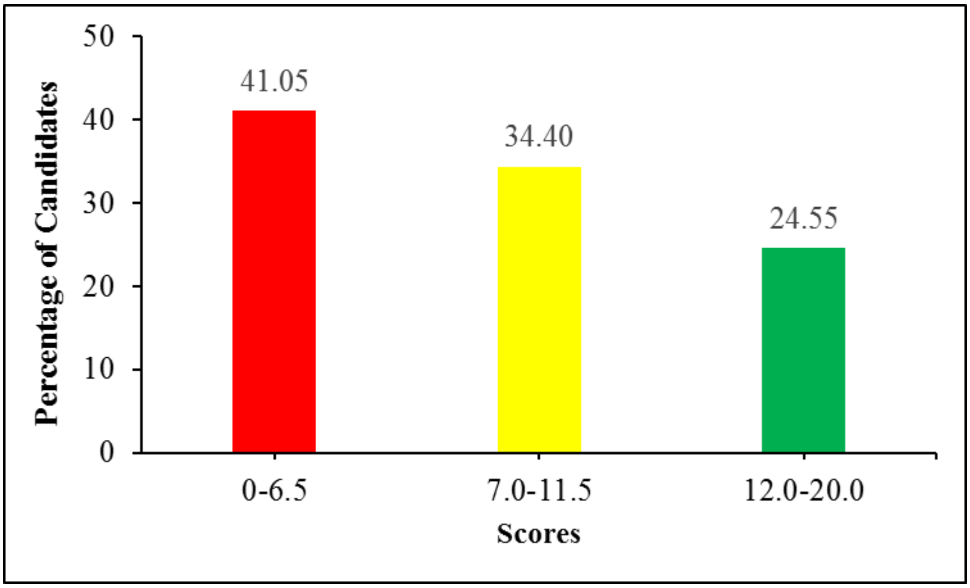


Figure 18: *The Candidates' Performance on Question 7 in Paper 2*

As shown in Figure 18, 1,090 (58.95%) candidates scored 7.0 marks or more, indicating an average performance on this question. Analysis of the responses shows that the candidates who performed well on this question had good competence in the concepts of differential equations. In part (a) (i), the candidates solved linear differential equations using the integrating factor. They found the integrating factor of the equation $\frac{dy}{dx} - \left(\frac{1}{x}\right)y = x^2$

and got $I = e^{\ln\left(\frac{1}{x}\right)} = \frac{1}{x}$. Using this integrating factor, they solved the differential equation and got the correct answer $\frac{y}{x} = \frac{x^2}{2} + c$. In part (a) (ii), the candidates solved the differential equation $\frac{dy}{dx} - x\tan(y-x) = 1$ by subtracting $y - x = t$ and its derivative $\frac{dy}{dx} = 1 + \frac{y}{x}$ to reduce the equation to the variable separable differential equation $\frac{dy}{dx} - x\tan t = 0$. Upon solving this equation, they arrived at the correct answer, $y = x + \sin^{-1}\left(Ae^{\frac{x^2}{2}}\right)$.

In part (b), the candidates found the first and second derivatives of the equation $x = A\cos(2t + B)$. They got $\frac{dx}{dt} = -2A\sin(2t + B)$ and $\frac{d^2x}{dt^2} = -4A\cos(2t + B)$, and upon substituting, they arrived at the required differential equation $\frac{d^2x}{dt^2} = -4x$. In part (c), the candidates applied the concept of differential equations to solve real-life problems concerning the motions of particles under gravity. They reduced the second-order differential equation of the given problem to a first-order differential equation using correct substitutions. They let $\frac{dx}{dt} = P$ and $\frac{d^2x}{dt^2} = \frac{dP}{dt}$, then they substituted these into $\frac{d^2x}{dt^2} = g - k\frac{dx}{dt}$ to get $\frac{dP}{dt} = g - kP$. They solved this equation as a linear differential equation $\frac{dP}{dt} + kP = g$ to get $P = \frac{g}{k} + ce^{-kt}$. They replaced P with $\frac{dx}{dt}$ to get $\frac{dx}{dt} = \frac{g}{k} + ce^{-kt}$, and upon solving this as a variable separable differential equation, they obtained $x = \frac{g}{k}t + \frac{g}{k^2}e^{-kt} + A$. Then, they substituted $x = 0$ and $t = 0$ into the equation to obtain $x = \frac{g}{k}t + \frac{g}{k^2}e^{-kt} - \frac{g}{k^2}$. In part (d), the candidates reduced the given equation $\left(2x-1\right)\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$ to the first-order variable separable differential equation $\left(2x-1\right)\frac{dP}{dx} - 2P = 0$ where $\frac{dy}{dx} = P$. They solved this equation to obtain $P = A(2x-1)$. Then replaced P with $\frac{dx}{dt}$ to get $\frac{dx}{dt} = A(2x-1)$ and then solved the equation using the concept of separating the variables to obtain $y = Ax^2 - Bx + C$. Thereafter, they substituted $x = 0$ and $y = 2$ into $y = Ax^2 - Ax + B$ to get $B = 2$ and then $x = 0$ and $\frac{dy}{dx} = 3$ into $\frac{dy}{dx} = A\left(2x-1\right)$ to get $A = -3$. At last they replaced A and B with their values to get $y = -3x^2 + 3x + 2$. Extract 17.1 provides a sample of correct responses from one of the candidates who demonstrated adequate skills in attempting this question.

7. a) i) $\frac{dy}{dx} - \frac{y}{x} = x^2$

Let $I \cdot F = e^{\int -\frac{1}{x} dx}$

$I \cdot F = e^{\ln x^{-1}}$

$I \cdot F = x^{-1}$

Multiply by I·F both sides

$x^{-1}\frac{dy}{dx} - \frac{y}{x}x^{-1} = x^2 x^{-1}$

$\frac{dy}{x\,dx} - \frac{y}{x^2} = x$

$\frac{d}{dx}(y/x) = x$

$d(y/x) = x\,dx$

Apply integral sign

$\int d(y/x) = \int x\,dx$

$\frac{y}{x} = \frac{x^2}{2} + c$

$\therefore \frac{y}{x} - \frac{x^2}{2} = c$

ii) $\frac{dy}{dx} - x\tan(y-x) = 1$ --- (i)

Let

$y - x = z$

$\frac{dy}{dx} - 1 = \frac{dz}{dx}$

$\frac{dy}{dx} = \frac{dz}{dx} + 1$ --- (ii)

Substitute eqn (ii) into into (i)

$\frac{dz}{dx} + 1 - x\tan z = 1$

$\frac{dz}{dx} = x\tan z$

7\. a) ii) $\frac{dz}{\tan z} = x\,dx$

$\int \frac{dz}{\tan z} = \int x\,dx$

$\ln \sin z = \frac{x^2}{2} + c$

$\ln \sin z = \frac{x^2}{2} + c$

$\ln \sin(y-x) = \frac{x^2}{2} + c$

$\therefore \ln \sin(y-x) - \frac{x^2}{2} = c$

b) $x = A\cos(2t+B)$ ---(i)

$\frac{dx}{dt} = -2A\sin(2t+B)$ ---(ii)

$\frac{d^2x}{dt^2} = -4A\cos(2t+B)$ ---(iii)

But $A\cos(2t+B) = x$

$\frac{d^2x}{dt^2} = -4x$

$\frac{d^2x}{dt^2} + 4x = 0$ or

$x'' + 4x = 0$

c) Given

$\frac{d^2x}{dt^2} = g - k\frac{dx}{dt}$

$\frac{d^2x}{dt^2} + k\frac{dx}{dt} = g$

7. g Let

$$\frac{dx}{dt} = P$$

$$\frac{d^2x}{dt} = \frac{dP}{dt}$$

$$\frac{dP}{dt} + kP = g$$

$$\frac{dP}{dt} = g - kP$$

$$\frac{dP}{g-kP} = dt$$

Apply integral sign to both sides

$$\int \frac{dP}{g-kP} = \int dt$$

$$-\frac{1}{k} \ln (g-kP) = t + c \quad ---(i)$$

$$\ln (g-kP)^{-1} = kt + kc$$

Let kc be D

$$\ln \left(\frac{1}{g-kP}\right) = kt + D$$

But $P = \frac{dx}{dt}$

Apply e both sides

$$\frac{1}{g-kP} = e^{(kt+D)}$$

$$g - kP = e^{-(kt+D)}$$

$$g - k\frac{dx}{dt} = e^{-(kt+D)}$$

$$k\frac{dx}{dt} = g - e^{-(kt+D)}$$

$$k\frac{dx}{dt} = g - e^{-(kt+D)} \quad ---- (ii)$$

7\. c) $\int k\,dx = \int (g - e^{-(kt+D)})\,dt$

$kx = gt - \int e^{-(kt+D)}\,dt$

Let

e^{-D} be constant E

$kx = gt - \int E e^{-kt}\,dt$

$kx = gt + \frac{E}{k} e^{-kt} + F$ - - - - (iii)

Consider eqn (ii)

$k\,\frac{dx}{dt} = g - e^{-(kt+D)}$

$0 = g - e^{-(0+D)}$

$e^{-D} = g$

But

$D = kc$

$e^{-kc} = g$

$-kc = \ln g$

$c = -\frac{1}{k} \ln g$ - - - - (iv)

Also

Consider eqn (iii)

$kx = gt + \frac{E}{k} e^{-kt} + F$

$0 = 0 + \frac{E}{k} e^{0} + F$

$F = -\frac{E}{k}$

But

$E = e^{-D}$

$e^{-D} = g$

$E = g$

$F = -\frac{g}{k}$ - - - - - - (v)

7\. c) function of x is given by

$$kx = gt + \frac{g}{k}e^{-kt} - \frac{g}{k}$$

$$x = \frac{gt}{k} + \frac{g}{k^2}e^{-kt} - \frac{g}{k^2}$$

$$x = \frac{1}{k}\left(gt + \frac{g}{k}e^{-kt} - \frac{g}{k}\right).$$

d) $(2x-1)\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$

Let $\frac{dy}{dx} = P$

$$\frac{d^2y}{dx^2} = \frac{dP}{dx}$$

$$(2x-1)\frac{dP}{dx} - 2P = 0$$

$$\frac{dP}{dx} = \frac{2P}{2x-1}$$

$$\frac{dP}{2P} = \frac{dx}{2x-1}$$

$$\frac{1}{2}\ln P = \frac{1}{2}\ln A(2x-1)$$

$$\ln P = \ln A(2x-1)$$

$$P = A(2x-1) \quad ----(i)$$

$$\frac{dy}{dx} = A(2x-1) \quad ----(ii)$$

$$dy = A(2x-1)\,dx$$

Apply integral sign both sides

$$\int dy = \int A(2x-1)\,dx$$

$$y = Ax^2 - Ax + B \quad ---(iii)$$

Consider eqn (ii) when $\frac{dy}{dx} = 3$, $x = 0$

7\. d) $3 = A(2\times0 - 1)$

$3 = -A$

$A = -3 \quad ----(iv)$

Also consider eqn (iii) when $y = 2$

$y = Ax^2 - Ax + B$

$2 = 0 - 0 + B$

$B = 2$

$\therefore y = -3x^2 + 3x + 2$

Extract 17.1: A sample of correct responses to question 7 in paper 2

In Extract 17.1, the candidate applied the correct concepts in formulating and solving differential equations and in applying differential equations to solve real-life problems related to the motion of objects under gravity.

On the other hand, analysis of the responses indicates that the candidates who failed to answer this question correctly lacked sufficient competence in formulating, solving, and applying differential equations to solve real-life problems. In part (a) (i), the candidates used an incorrect method to solve the equation. For instance, instead of using the integrating factor, they applied the concept of a homogeneous differential equation. In part (a) (ii), the candidate failed to recognise the appropriate method to use in solving the differential equation $\frac{dy}{dx} - x\tan(y-x) = 1$. In part (b), the candidate failed to apply the correct concept of differentiation to formulate the differential equation. To differentiate the equation $x = A\cos(2t+B)$, they wrote $x' = \cos(2t+B) - 2A\sin(2t+B)$ and $x'' = -2\sin(2t+B) - 2\sin(2t+B) - 4A\cos(2t+B)$. This misconception led them to wrong answers.

In part (c), most candidates attempted to apply the concept of homogeneous second-order differential equations, which did not help them to solve the problem. For example, using the equation $\frac{d^2x}{dx^2} - g - k\frac{dx}{dt}$, they substituted $m = \frac{dx}{dt}$ and obtained $m^2 + km - g = 0$. Then, they solved this equation and got $m = \frac{-k \pm \sqrt{k^2 + 4g}}{2}$ and concluded distance, $x = \frac{-k \pm \sqrt{k^2 + 4g}}{2}$. In part (d), the candidates failed to apply the correct concepts in solving the equation $(2x-1)\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$. For instance, they correctly let $\frac{dy}{dx} = P$ and then reduced the equation into first order $(2x-1)\frac{dP}{dx} - 2P = 0$. After that, they incorrectly applied the concept of chain rule $\frac{dy}{dx} = \frac{dy}{dP}.\frac{dP}{dx}$. Using this misconception, they ended up with an incorrect answer, $\frac{dx}{dt} = \frac{2x-1}{2}$.

Extract 17.2 shows a sample of incorrect responses from a candidate who answered the question incorrectly.

7b) Given

$$X = A\cos(2t+B) \quad \text{---} (i)$$

$$X' = \cos(2t+B) + -2\sin(2t+B)A \quad \text{---} (ii)$$

$$= \cos(2t+B) - 2A\sin(2t+B).$$

$$X'' = -2\sin(2t+B) - 2\sin(2t+B) - 4A\cos(2t+B) \quad \text{---} (iii)$$

$$= -2\sin(2t+B) - 2\sin(2t+B) - 4A\cos(2t+B)$$

$$= -4\sin(2t+B) - 4A\cos(2t+B).$$

$$= -4(\sin 2t+B) + A\cos 2t+B.$$

But

$$A\cos(2t+B) = X.$$

Now,

$$X'' = -4(\sin 2t + B + X). \quad \text{--- eqn (iv)}$$

from equation (ii)

$$X' = \cos(2t+B) - 2A\sin(2t+B).$$

$$2A\sin(2t+B) = \cos(2t+B) - X'.$$

Divide by 2A both sides to get $\sin(2t+B)$

$$\sin(2t+B) = \frac{\cos(2t+B) - X'}{2A}$$

7 5) Now, from eqn (iv)

$$x'' = -4(\sin 2t + B + x).$$

$$\sin 2t + B = \frac{\cos(2t+B) - x'}{2A}$$

Substitute the value of $\sin 2t+B$ to eqn (iv)

$$x'' = -4\left(\frac{\cos(2t+B) - x'}{2A} + x\right).$$

$$x'' = -4\left(\frac{\cos(2t+B) - x' + 2Ax}{2A}\right).$$

$$x'' = -4\left(\frac{\cos(2t+B) - x'}{2A} + \frac{2Ax}{2A}\right).$$

$$x'' = -4\left(\frac{\cos(2t+B) - x'}{2A} + 2x\right). \quad \text{--- eqn (v)}$$

from equation (i)

$$x = A\cos(2t+B)$$

$$A = \frac{x}{\cos(2t+B)}$$

Substitute the value of A to eqn (v)

7 b)

$$x'' = -4\left(\frac{\cos(2t+\beta) - x'}{2\left(\frac{x}{\cos(2t+\beta)}\right)} + x\right)$$

$$x'' = -4\left(\frac{\cos(2t+\beta) - x'}{\frac{2x}{\cos(2t+\beta)}} + x\right).$$

$$x'' = \left(\frac{\cos 2t+\beta - x'}{\frac{2x}{\cos(2t+\beta)}} + x\right)$$

$$x'' = -4\left(\frac{-x'}{2x} + \frac{x}{1}\right).$$

$$x'' = -4\left(\frac{-x' + 2x^2}{2x}\right).$$

$$x'' = -4\left(\frac{2x^2 - x'}{2x}\right).$$

$$x'' = -4x - x'/x.$$

7y Given

$$\frac{d^2x}{dt^2} = g - k\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} = g$$

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} - g = 0.$$

let $\frac{dx}{dt} = m$

$$m^2 + km - g = 0.$$

from a General formula

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where $a = 1$

$b = k.$

$c = -g.$

$$m = \frac{-k \pm \sqrt{k^2 - (4 \times 1 \times -g)}}{2}$$

$$m = \frac{-k \pm \sqrt{k^2 + 4g}}{2}$$

7 g) Then,

$$m = \frac{-k \pm \sqrt{k^2+49}}{2}$$

The distance $x = \frac{-k \pm \sqrt{k^2+49}}{2}$

d) Given

$$(2x-1)\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0.$$

let

$$p = dy/dx.$$

$$\frac{d}{dx}(p) = \frac{d^2y}{dx^2}$$

Substitute to the given equation

$$(2x-1)\left(\frac{dp}{dx}\right) - 2(p) = 0.$$

$$(2x-1)\frac{dp}{dx} - 2p = 0.$$

But

$$\frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx}.$$

7d) Then,

$$\frac{dy}{dx} = \frac{dp}{dx} \times \frac{dy}{dp}$$

from $(2x-1)\frac{dp}{dx} - 2p = 0.$

$$(2x-1)\frac{dp}{dx} = 2p.$$

Divide by $(2x-1)$ both side.

$$\frac{dp}{dx} = \frac{2p}{(2x-1)}$$

Now,

$$\frac{dy}{dx} = p.$$

$$\frac{dp}{dx} = \frac{2p}{2x-1}$$

dy/dp = ? — Required.

Then,

$$p = \left(\frac{2p}{2x-1}\right) \times \frac{dy}{dp}.$$

$$\frac{dy}{dp} = p \cdot \frac{2x-1}{2p}.$$

$$\frac{dy}{dp} = \frac{p(2x-1)}{2p}$$

7d)

$$dy/dp = \frac{2x-1}{2}.$$

The value of $dy/dp = \frac{2x-1}{2}$

Extract 17.2: A sample of incorrect responses to question 7 of paper 2

In Extract 17.2, the candidate failed to apply the correct techniques of differentiation to formulate the required differential equation, used incorrect methods to solve first-order differential equations, and did not apply the appropriate techniques for reducing second-order differential equations to first-order.

2.2.8 Question 8: Coordinate Geometry II

The question examined the candidate's ability to identify conic sections, derive their equations, express the equations of conic sections in standard form, determine the equations of tangents to conic sections, and sketch the conic sections on the xy-plane. The question was as follows:

(a) *An equilateral triangle is inscribed in a parabola* $y^2 = 4ax$, *whose vertex is at the vertex of the parabola. Find the length of the side of the triangle.*

(b) *If two tangent lines to the ellipse* $9x^2 + 4y^2 = 36$ *intersect the y-axis at point* $P(0,6)$, *find the point of tangency.*

(c) *Find the equation of a circle that passes through points* $A(1,2)$, $B(2,5)$, *and* $C(-3,4)$. *Write the equation in the form* $x^2 + y^2 + 2gx + 2fy + c = 0$, *where f, g, and c are constants.*

(d) *What is the length of the tangents from point* $(6,2)$ *to a circle with centre* $(2,1)$ *and radius* 2 *?*

(e) *Write the equation of ellipse* $18x^2 + 12y^2 - 144x + 48y + 120 = 0$ *in a standard form.*

(f) *Sketch the ellipse having these characteristics: a central ellipse, foci at* $(\pm 4,0)$, *and vertices at* $(\pm 5,0)$.

The analysis of data revealed that 4,626 (100%) candidates attempted the question. Of these, 1,989 (43.00%) scored from 0 to 6.5 marks, 2,129 (46.02%) scored from 7.0 to 11.5 marks, and 508 (10.98%) candidates scored from 12.0 to 20.0 marks. The candidates' performance on this question was generally average, as 57.00 percent passed. Figure 19 provides a summary of the candidates' performance on this question.

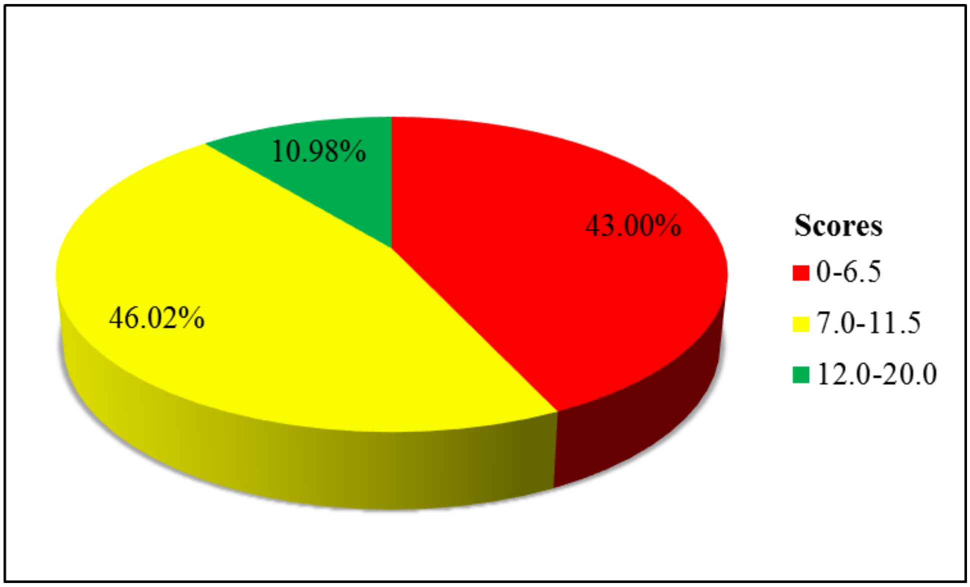
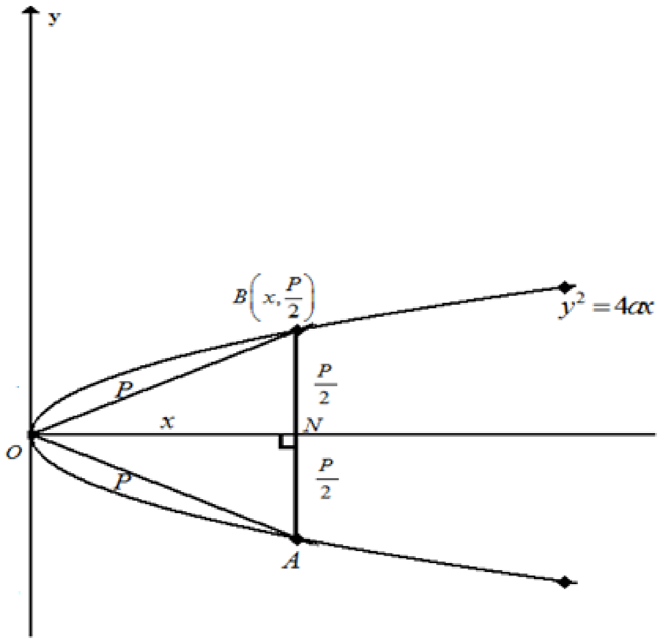


Figure 19: *The Candidates' Performance in Question 8 of Paper 2*

The data shows that 9 (0.19%) candidates scored full marks as they correctly responded to the question. In part (a), the candidates drew a simple sketch to determine how the triangle is inscribed in a parabola $y^2 = 4ax$, as shown in the figure below.



They applied the concept of Pythagoras's Theorem to obtain the relation $P^2 = x^2 + \frac{P^2}{4}$. Then, they substituted the coordinates $\left(x, \frac{p}{2}\right)$ into the equation $y^2 = 4ax$ to solve for x, from which they obtained $x = \frac{p^2}{16a}$. Upon substituting this into $P^2 = x^2 + \frac{P^2}{4}$, they obtained $P = 8\sqrt{3}a$ units, which is the required length of the side of the triangle.

In part (b), the candidates let the equations of the tangents be of the form $y = mx + c$. Then, they substituted the coordinates $(0, 6)$ into the equation to get $c = 6$. Further, they substituted $y = mx + 6$ into the equation of the ellipse $9x^2 + 4y^2 = 36$ to formulate the quadratic equation $(9 + 4m^2)x^2 + 48mx + 108 = 0$. Thereafter, they applied the condition $b^2 = 4ac$, where $a = 9 + 4m^2$, $b = 48m$, and $c = 108$, to obtain $m = \pm\frac{3\sqrt{3}}{2}$. Then, they solved the equations $y = \pm\frac{3\sqrt{3}}{2}x + 6$ and $9x^2 + 4y^2 = 36$ simultaneously to get the required points of tangency, $(x, y) = \left(-\sqrt{3}, \frac{3}{2}\right)$ and $(x, y) = \left(\sqrt{3}, \frac{3}{2}\right)$.

In part (c), the candidates recalled the general form of the equation of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$, and then they substituted the given points $A(1, 2)$, $B(2, 5)$, and $C(-3, 4)$ into this equation to obtain three simultaneous equations: $2g + 4f + c = -5$, $4g + 10f + c = -29$, and $-6g + 8f + c = -25$. They solved these equations to get $g = \frac{3}{7}$, $f = -\frac{29}{7}$, and $c = \frac{75}{7}$, and upon substituting these values into the general equation of a circle, they obtained the correct equation $x^2 + y^2 + \frac{6}{7}x - \frac{58}{7}y = \frac{75}{7}$. In part (d), the candidates determined the equation of a circle with centre $(2, 1)$

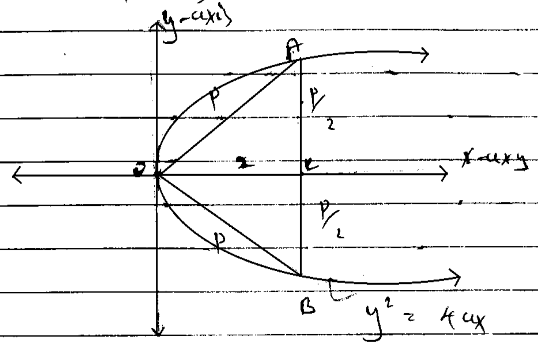
and radius 2 by taking $(x-2)^2+(y-1)^2=4$. Then, they simplified it to get $x^2+y^2-4x-2y+1=0$. To find the length of the tangent from the point $(6,2)$, they recalled the formula $L=\sqrt{x_1{}^2+y_1{}^2-4x_1{}^2-2y_1{}^2+1}$ and then substituted $(x_1,y_1)=(6,2)$ to get $L=\sqrt{13}$, which is the correct answer.

In part (e), the candidates rewrote the equation $18x^2+12y^2-144x+48y+120=0$ in the form $\frac{(x-x_1)^2}{a}+\frac{(y-y_1)^2}{b}=1$ by applying the completing the square technique. They started by collecting like terms to get $18x^2-144x+12y^2+48y=-120$ and then completed the square to obtain the required form $\frac{(x-4)^2}{12}+\frac{(y+2)^2}{18}=1$. In part (f), the candidates used the given vertices $(-5,0)$ and $(5,0)$ and the y-intercepts to sketch the graph of the ellipse. To find the y-intercepts, the candidates first determined the equation of the ellipse. They obtained the equation of the ellipse as $\frac{x^2}{25}+\frac{y^2}{9}=1$, and from this they got y-intercepts $(0,-3)$ and $(0,3)$. Extract 18.1 shows a sample of correct responses from a candidate who answered the question correctly.

8 a) Soln

Required length of a side of the triangle

consider the following sketch.



let the length of one side of triangle be p

from equation of parabola

$y^2 = 4ax$

at point $(x, p/2)$

$\left(\frac{p}{2}\right)^2 = 4ax$

$\frac{p^2}{4} = 4ax$

$x = \frac{p^2}{16a}$

Considering Δ OCA

by pythagoras theorem

$(\overline{OA})^2 = (\overline{OC}) + (\overline{AC})^2$

$p^2 = \left(\frac{p^2}{16a}\right)^2 + \left(\frac{p}{2}\right)^2$

$p^2 = \frac{p^4}{256a^2} + \frac{p^2}{4}$

8

$$1 = \frac{p^2}{256a^2} + \frac{1}{4}$$

$$\frac{3}{4} = \frac{p^2}{256a^2}$$

$$\frac{3}{1} = \frac{p^2}{64a^2}$$

$$p^2 = 192a^2$$

$$p = 8\sqrt{3}\,a \text{ units}$$

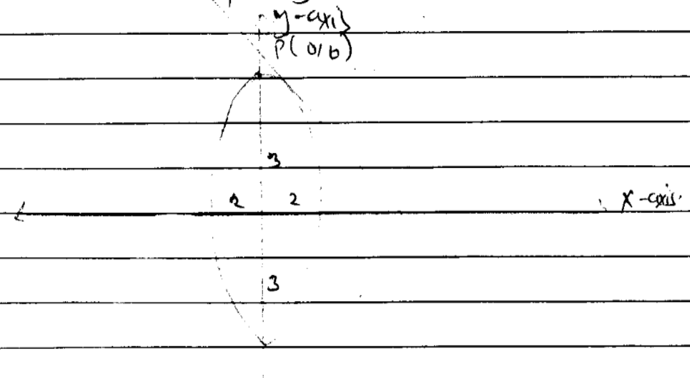
∴ The length of one side of triangle is $8\sqrt{3}\,a$ units.

54) Soln

Given ellipse: $9x^2 + 4y^2 = 36$

Required the point of tangency

consider the following sketch:



In standard form the ellipse is given as

~~x^2+~~ $\frac{x^2}{4} + \frac{y^2}{9} = 1$

8 from the equation of ~~tangency~~ ellipse

$9x^2 + 4y^2 = 36$

differentiate w.r.t x

$18x + 8y\frac{dy}{dx} = 0$

$8y\frac{dy}{dx} = -18x$

$\frac{dy}{dx} = \frac{-18x}{8y}$

$\frac{dy}{dx} = \frac{-9x}{4y}$

m = slope of equation of tangent

by equation of straight line.

$y = m(x - x_1) + y_1$

at point $(0, 6)$

$y = m(x - 0) + 6$

$y = mx + 6$

$9x^2 + 4y^2 = 36$

$9x^2 + 4(mx+6)^2 = 36$

$9x^2 + 4(m^2x^2 + 2mx6 + 36) = 36$

$9x^2 + 4m^2x^2 + 48mx + 144 = 36$

$(9 + 4m^2)x^2 + 48mx + 108 = 0$

at point of tangency discriminant = 0

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{(-48m) \pm \sqrt{(48m)^2 - 4 \times 108 \times (9 + 4m^2)}}{2(9+4m^2)}$

$(48m)^2 - 4 \times 108 \times (9 + 4m^2) = 0$

$2304m^2 - 3888 - 1728m^2 = 0$

$576m^2 = 3888$

$m = \frac{3\sqrt{3}}{2}$

8	$m = \frac{3\sqrt{3}}{2}$
	$x = \frac{-48m}{18 + 8m^2}$
	at $m = \frac{3\sqrt{3}}{2}$
	$x = -\sqrt{3}$.
	but from
	$y = mx + 6$
	$y = \frac{3\sqrt{3}}{2} \times -\sqrt{3} + 6$
	$y = \frac{3}{2}$
	∴ Point of tangency $= \left(-\sqrt{3}, \frac{3}{2}\right)$
	e) Sln
	Given points
	A(1, 2) B(2, 5) and C(-3, 4)
	from standard equation of a circle
	$x^2 + y^2 + 2gx + 2fy + c = 0$
	at (1, 2)
	$(1)^2 + (2)^2 + 2g(1) + 2f(2) + c = 0$
	$1 + 4 + 2g + 4f + c = 0$
	$2g + 4f + c = -5$ --- (i)
	at point (2, 5)
	$(2)^2 + (5)^2 + 2g(2) + 2f(5) + c = 0$
	$4 + 25 + 4g + 10f + c = 0$
	$4g + 10f + c = -29$ --- (ii)

& at point $(-3, 4)$

$$(-3)^2 + (4)^2 + 2g(-3) + 2f(4) + c = 0$$

$$9 + 16 - 6g + 8f + c = 0$$

$$25 - 6g + 8f + c = 0$$

$$-6g + 8f + c = -25 \quad \text{--- (iii)}$$

by solving eqn (i), (ii) and (iii) simultaneously,

$$g = \frac{3}{7} \quad f = \frac{-29}{7} \quad c = \frac{75}{7}$$

from standard equation of circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$x^2 + y^2 + 2\left(\frac{3}{7}\right)x + 2\left(\frac{-29}{7}\right) + \frac{75}{7} = 0$$

$$x^2 + y^2 + \frac{6x}{7} - \frac{58}{7}y + \frac{75}{7} = 0$$

$$7x^2 + 7y^2 + 6x - 58y + 75 = 0$$

hence equation of a circle

$$7x^2 + 7y^2 + 6x - 58y + 75 = 0$$

8 f)

Sln

Required a sketch of an ellipse

foci $(\pm 4, 0)$

vertices $(\pm 5, 0)$

Its lies on x - axis

$ae = \pm 4$

but $a = \pm 5$

$5e = 4$

$e = \frac{4}{5}$

directrices $= x \pm$

$x = \pm \frac{a}{e}$

$= \pm \frac{5}{\frac{4}{5}}$

directrices $= x = \pm \frac{25}{4}$

Since $a > b$

$b^2 = a^2(1 - e^2)$

$b^2 = 25\left[1 - \frac{16}{25}\right]$

$b^2 = 9$

$b = 3$

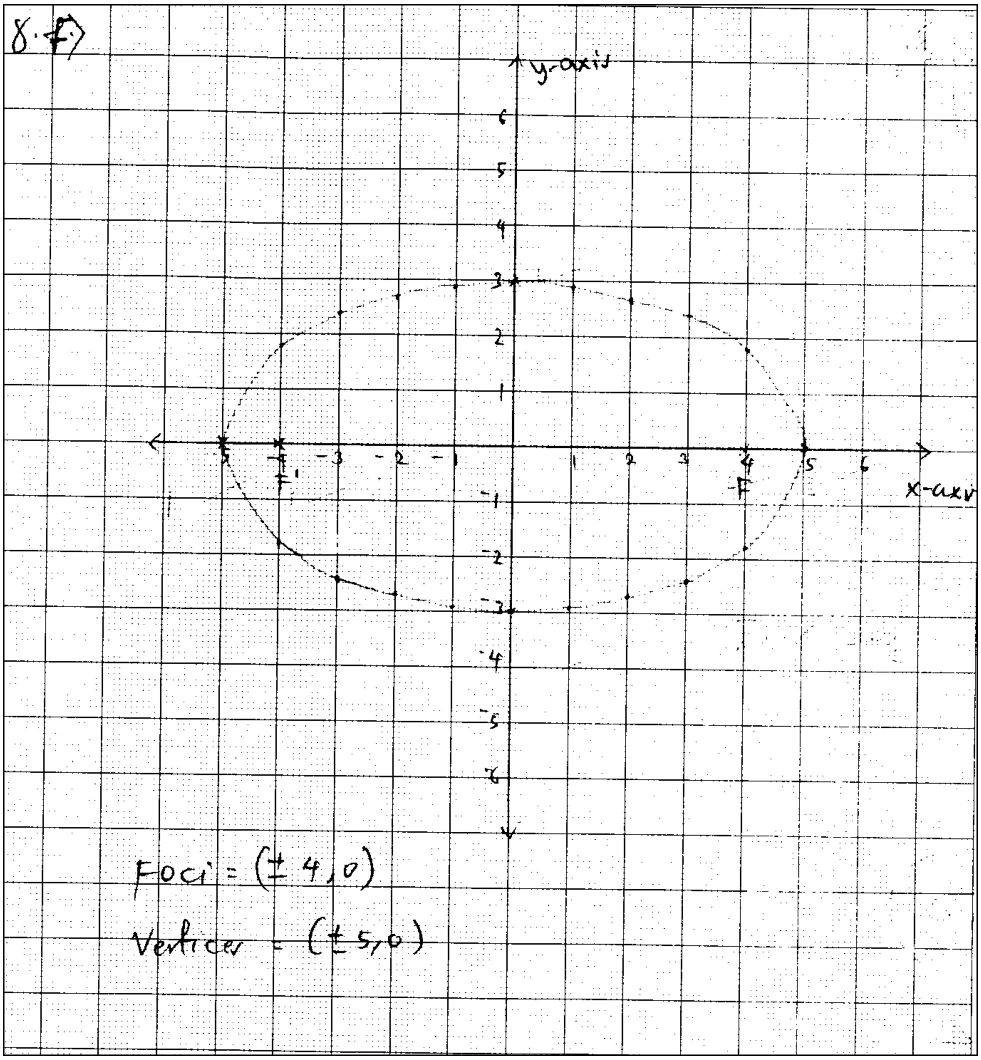
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

at $\frac{x^2}{25} + \frac{y^2}{9} = 1$

x-intercept $y = 0$

$\frac{x^2}{16} = 1$

$x^2 = 25$ $x = \pm 5$



Extract 18.1: A sample of correct responses to question 8 in paper 2

In Extract 18.1, the candidate correctly determined the length of the side of the triangle and the length of tangents to the conic section in parts (a) and (d), respectively. He/she expressed the equations of conic sections (a circle and an ellipse) in standard form in parts (c) and (e). The candidate also determined the point of tangency to the conic and sketched the conic sections on the xy-plane in parts (b) and (f), respectively.

In spite of the average performance, 182 (3.93%) candidates scored zero due to inadequate knowledge and skills on the concepts of conic sections. In part (a), the candidates failed to interpret the question. Instead of finding the length of the side of the triangle, they calculated the distance between the points (x, y) and $(a, 0)$ using the distance formula. They wrote $length = 2\sqrt{(x-a)^2 + y^2}$ units. In part (b), the candidates applied an incorrect concept to find the equation of the tangent by differentiating the equation of the ellipse $9x^2 + 4y^2 = 36$ to get $\frac{dy}{dx} = -\frac{ax}{4y}$ and then substituted $(0, 6)$ to get $\frac{dy}{dx} = 0$. Using the substituted value, the candidates ended up with incorrect equations for the tangents and points of tangency. In part (c), the candidates correctly recalled the general form of the equation of a circle, $x^2 + y^2 + 2gx + 2fy + c = 0$, and substituted the given points $A(1, 2)$, $B(2, 5)$ and $C(-3, 5)$ into this equation, but they made computational errors. Similarly, others failed to obtain the equation of the circle because they used only two equations to solve a system of simultaneous equations containing three unknowns: f, g, and c.

In part (d), the candidates correctly recalled and applied the general equation of a circle and the distance formula, but they failed to obtain the correct length of the tangents due to computational errors. In part (e), the candidates failed to apply the completing the square method to express the given equation $18x^2 + 12y^2 - 144x + 48y + 120 = 0$ in standard form. Additionally, some made computational errors, such as writing $8(x^2 - 8x) + 12(y^2 + 4y) + 120 = 0$ after $18x^2 - 144x + 12y^2 + 48y + 120 = 0$. In part (f), some candidates failed to sketch the ellipse because they were unable to find the correct points where it intersects the y-axis. Extract 18.2 shows a sample of incorrect responses from a candidate who answered the question incorrectly.

8 a) but $\bar{PS} = \bar{SQ}$

Length $\bar{PQ} = 2\left(\sqrt{(x-a)^2+y^2}\right)$

Length $\bar{PQ} = 2\sqrt{(x-a)^2+y^2}$.

Length of the side of triangle;-

$\bar{PQ} = 2\sqrt{(x-a)^2+y^2}$ units.

$\therefore$ Length of side of triangle is $2\sqrt{(x-a)^2+y^2}$ units.

b) Solution.

Equation of tangents is given by;-

$9x^2 + 4y^2 = 36$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

$\frac{d}{dx}\left(\frac{x^2}{4}\right) + \frac{d}{dx}\left(\frac{y^2}{9}\right) = 0.$

$\frac{2x}{4} + \frac{2y}{9}\frac{dy}{dx} = 0$

$\frac{2y}{9}\frac{dy}{dx} = -\frac{2x}{4}$

$\frac{dy}{dx} = -\frac{x}{2} \div \frac{2y}{9}$

$\frac{dy}{dx} = -\frac{9x}{4y}$ at point $(0,6)$

$\frac{dy}{dx} = 0.$

8 b)

$m = \frac{y_{*} - y_0}{x - x_0}$

but, $(x_0, y_0) = (0, 6)$.

$0 = \frac{y-6}{x-0}$

$0 = \frac{y-6}{x}$

$y - 6 = 0$.

$y = 6$.

$9x^2 + 4y^2 = 36$.

$9x^2 + 4(6)^2 = 36$.

$9x^2 + 144 = 36$.

$9x^2 = -108$

$x^2 = -12$.

Extract 18.2: A sample of incorrect responses to question 8 of paper 2

In Extract 18.2, the candidate applied an irrelevant formula for the distance between two points to find the length of the side of the triangle, used an incorrect concept to find the equation of the tangent, and applied a wrong concept to determine the points of tangency.

3.0 ANALYSIS OF CANDIDATES' PERFORMANCE IN EACH TOPIC

The 2025 Advanced Mathematics examination covered a total of 18 topics, with 10 drawn from Advanced Mathematics 1 and 8 from Advanced Mathematics 2. Each question was based on a different topic. The topics examined in Advanced Mathematics 1 included: *Calculating Devices*, *Hyperbolic Functions*, *Linear Programming*, *Statistics*, *Sets*, *Functions*, *Numerical Methods*, *Coordinate Geometry I*, *Integration*, and *Differentiation*. In Advanced Mathematics 2, the topics assessed were: *Probability*, *Logic*, *Vectors*, *Complex Numbers*, *Trigonometry*, *Algebra*, *Differential Equations*, and *Coordinate Geometry II*.

An analysis of the candidates' performance across these topics reveals that fourteen (14) topics were well performed, three (3) showed average performance, and one (1) was poorly performed. The topic with weak performance in the examination was *Integration* (21.76%). The poor performance was attributed to a lack of sufficient competence in the concept of integration and its application, as well as incorrect interpretation of the question. The topics with good performance in the examination were: *Logic* (95.58%), *Functions* (94.59%), *Linear Programming* (90.86%), *Coordinate Geometry I* (88.95%), *Statistics* (84.83%), *Numerical Methods* (84.51%), *Sets* (78.25%), *Trigonometry* (75.85%), *Algebra* (75.53%), *Vectors* (67.47%), *Calculating Devices* (66.62%), *Probability* (65.07%), *Hyperbolic Functions* (64.83%), and *Differentiation* (64.38%). The three topics with average performance were: *Differential Equations* (58.95%), *Coordinate Geometry II* (57.00%) and *Complex Numbers* (45.90%). The good and average performance in these topics was attributed to the candidates' competence in the following areas:

(a) Using a non-programmable calculator to compute the values of different expressions correctly to a given number of decimal places and significant figures.
(b) Using the correct statistical formulae to determine the measures of central tendency and dispersion.
(c) Representing sets on a number line and using Venn diagrams to solve related problems.
(d) Using the laws of the algebra of propositions to simplify given logical expressions and using truth tables to test the validity of arguments.
(e) Understanding the properties of a probability density function and applying them to determine whether given functions are probability density functions or not.
(f) Calculating the mean and standard deviation of a normal distribution by applying the correct formulae.
(g) Differentiating functions and applying differentiation to solve real-life problems.
(h) Applying the concept of the cross product to determine the area of a parallelogram.
(i) Calculating the perpendicular distance from a point to a line and finding the length of a tangent to a circle from that point.
(j) Applying basic hyperbolic identities and definitions to prove identities and differentiate hyperbolic expressions.

(k) Formulating and solving transportation linear programming problems graphically.
(l) Drawing graphs of polynomial and rational functions and using them to determine domains and ranges.
(m) Deriving the secant formula and using both the Secant and Newton-Raphson methods to find the roots of equations.
(n) Applying the laws of logarithms to solve equations involving logarithmic expressions.
(o) Using Cramer's Rule to solve systems of simultaneous equations.
(p) Using mathematical induction to prove mathematical statements.
(q) Solving first-order differential equations using the integrating factor method, forming second-order differential equations, and determining solutions when second-order equations are reduced to first-order.

4.0 CONCLUSION AND RECOMMENDATIONS

4.1 Conclusion

Generally, in ACSEE 2025, the performance in Advanced Mathematics declined by 0.19 percent, with 97.51 percent of candidates passing the examination when compared to 97.70 percent in ACSEE 2024. Despite this slight decrease, the analysis shows that candidates performed well in *Logic* (95.58%), while their performance in *Integration* (21.76%) was weak. The poor performance in *Integration* was attributed to insufficient competence in the concept and its application.

4.2 Recommendations

To improve candidates' performance in the Advanced Mathematics examination, the following recommendations are given:

(a) Teachers should connect new content to students' prior knowledge when teaching complex sections of certain topics. For example, when demonstrating how to determine the Cartesian equation of the locus of complex numbers, they should begin by revising fundamental concepts such as modulus, argument, and the Argand diagram.

(b) In the topic of Integration, teachers should guide students;
   (i) to evaluate the definite integral of polynomial, trigonometric, exponential, logarithmic, and rational functions.
   (ii) through group discussion methods to find the area between two curves using integration.

(iii) on how to find the volume of solid objects formed by rotating simple curves about the x-axis or the y-axis through group discussions.

(c) In the topic of Coordinate Geometry II teachers should

(i) emphasise the use of diagrams; that is, have students sketch each conic section and label it appropriately. Encourage students to connect conic sections to real-life examples (e.g., satellite dishes, orbits, etc.) to enhance interest and understanding.

(ii) teach students how to identify the types of conic sections from a general equation. They should also demonstrate how to complete the square to convert the general equation into its standard form.

(d) During the teaching and learning process, teachers should address common mistakes done by many candidates, such as

(i) when finding the n^{th} roots of complex numbers using De Moivre's Theorem, forgetting to divide the angle by n.

(ii) in solving and proving trigonometric identities involving inverse trigonometric functions and trigonometric equations, using calculators to calculate angles directly.

(e) Teachers should emphasise the distinction between Cramer's Rule (determinant method) and the inverse matrix method when solving simultaneous equations.

(f) To increase students' participation and capability, teachers should encourage group work and discussion when solving problems, allowing students to work in pairs or small groups.

Appendix I

Analysis of Candidates' Performance on Each Topic for ACSEE 2025 Advanced Mathematics Examinations

S/N	Topic	Question Number	Percentage of Candidates who Passed	Remarks
1.	Logic	2 (P_2)	95.58	Good
2.	Functions	6 (P_1)	94.59	Good
3.	Linear Programming	3 (P_1)	90.86	Good
4.	Coordinate Geometry I	8 (P_1)	88.95	Good
5.	Statistics	4 (P_1)	84.83	Good
6.	Numerical Methods	7 (P_1)	84.51	Good
7.	Sets	5 (P_1)	78.25	Good
8.	Trigonometry	5 (P_2)	75.85	Good
9.	Algebra	6 (P_2)	75.53	Good
10.	Vectors	3 (P_2)	67.47	Good
11.	Calculating Devices	1 (P_1)	66.62	Good
12.	Probability	1 (P_2)	65.07	Good
13.	Hyperbolic Function	2 (P_1)	64.83	Good
14.	Differentiation	10 (P_1)	64.38	Good
15.	Differential Equations	7 (P_2)	58.95	Average
16.	Coordinate Geometry II	8 (P_2)	57.00	Average
17.	Complex Numbers	4 (P_2)	45.90	Average
18.	Integration	9 (P_1)	21.76	Weak

Appendix II

Analysis of Candidates' Performance in Each Topic in the 2024 and 2025 Advanced Mathematics Examinations

S/N	Topic	Question Number	2024		2025	
			The Percentage of Candidates who Passed	Remarks	The Percentage of Candidates who Passed	Remarks
1.	Logic	2	88.14	Good	95.58	Good
2.	Functions	6	98.65	Good	94.59	Good
3.	Linear Programming	3	83.11	Good	90.86	Good
4.	Coordinate Geometry I	8	68.17	Good	88.95	Good
5.	Statistics	4	92.91	Good	84.83	Good
6.	Numerical Methods	7	88.31	Good	84.51	Good
7.	Sets	5	98.04	Good	78.25	Good
8.	Trigonometry	5	90.08	Good	75.85	Good
9.	Algebra	6	75.45	Good	75.53	Good
10.	Vectors	3	88.53	Good	67.47	Good
11.	Calculating Devices	1	84.48	Good	66.62	Good
12.	Probability	1	24.93	Weak	65.07	Good
13.	Hyperbolic Function	2	88.75	Good	64.83	Good
14.	Differentiation	10	64.79	Good	64.38	Good
15.	Differential Equations	7	70.18	Good	58.95	Average
16.	Coordinate Geometry II	8	68.68	Good	57.00	Average
17.	Complex Numbers	4	76.40	Good	45.90	Average
18.	Integration	9	57.27	Average	21.76	Weak