THE UNITED REPUBLIC OF TANZANIA
MINISTRY OF EDUCATION, SCIENCE AND TECHNOLOGY
NATIONAL EXAMINATIONS COUNCIL OF TANZANIA

CANDIDATES' ITEM RESPONSE ANALYSIS REPORT ON THE ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION (ACSEE) 2022

ADVANCED MATHEMATICS

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142 ADVANCED MATHEMATICS

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FOREWORD

This report on the Candidates' Item Response Analysis (CIRA) in Advanced Mathematics for the Advanced Certificate of Secondary Education Examination (ACSEE) 2022 was prepared and issued by the National Examinations Council of Tanzania. The report is based on the performance of candidates with the aim of providing feedback about the candidates' competence in the Advanced Mathematics subject to students, teachers, parents, education administrators, school managers, policy makers and the general public.

The analysis of the candidates' responses was done in order to determine the candidates' strengths and weaknesses in application of what they had learned. The analysis shows what the secondary education system was able to provide for students during their two years of Advanced Level of secondary education.

The ACSEE 2022 Advanced Mathematics was well performed; 13,105 (97.89%) out of 13,449 candidates passed the examination. The good performance was observed in 14 out of 18 topics which were examined. The candidates performed well on the topics of Logic, Coordinate Geometry II, Trigonometry, Sets, Linear Programming, Statistics, Algebra, Calculating Devices, Vectors, Hyperbolic Functions, Numerical Methods, Differential Equations, Complex Numbers and Functions.

The candidates had average performance on the topics of Probability, Coordinate geometry I and Differentiation and they demonstrated poor performance on the topic of Integration. The candidates′ inability to use the method of integration by parts to integrate the product of algebraic and exponential functions and the use of partial fractions to evaluate the definite integral both contributed to their poor performance.

The National Examinations Council of Tanzania believes that the feedback provided in this report will help stakeholders in deciding on the most appropriate strategies of improving the future performance on this subject.

The Council would like to express its gratitude to all individuals, including examination officers who wrote this report.

Athumani S. Amasi

EXECUTIVE SECRETARY

1.0 INTRODUCTION

This report is meant to informs all education stakeholders on candidates' items responses in Advanced Mathematics subject in the Advanced Certificate of Secondary Education Examination (ACSEE) 2022.

Specifically, the report analyses the candidates' responses in all the topics examined in the Advanced Mathematics syllabus. The Advanced Mathematics examination had two papers: paper 1 and paper 2. In paper 1, there were ten (10) compulsory questions where each question carried ten (10) marks. Paper 2 consisted of sections A and B. Where section A comprised four (4) compulsory questions, where each question carried fifteen (15) marks. Section B comprised four (4) optional questions from which the candidates were required to answer any two questions and each carried twenty (20) marks.

The analysis of the candidates' responses involved short explanations on the requirement of each question and its performance is presented in section 2.0. Figures and charts have been used to summarize the performance. Factors that contributed to good, average and weak performance in each question have been illustrated using samples of candidates' responses which are inserted as extracts.

The analysis of candidates' responses in each topic has been done to identify topics with good, average and weak performance. Three colours have been used to indicate performance whereby green colour stands for good performance, yellow colour for average and red colour for weak performance. The percentage boundaries 0 – 34, 35 – 59 and 60 – 100 are used to represent weak, average and good performance respectively.

In 2022, a total of 13,449 candidates sat for the Advanced Mathematics Examination, out of whom 13,105(97.89%) candidates passed. However, the results of 61 candidates were withheld due to administrative reasons. In comparison, the performance in 2022 was better than that of 2021 where 12,706 (93.49%) candidates passed. This represents an increase of 4.4 per cent of the candidates' performance. The percentages of candidates' performance in Advanced Mathematics examination in two consecutive years are presented in Figure 1.

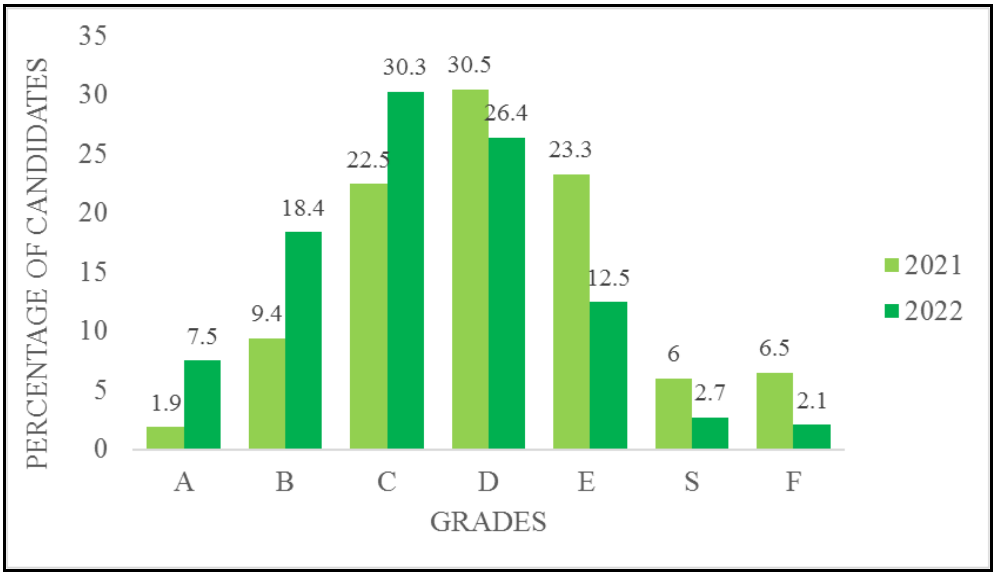


Figure 1: *Overall Performance in the 2021 and 2022 Advanced Mathematics Examinations*

In Figure 1, the trend shows a rise of the quality of performance in 2022 as compared to 2021. For example, the number of candidates who scored grade F decreased by 4.4% in 2022, while grade C increased by 7.8% and grade B increased by 9%. On the other hand, the number of candidates who scored grade A in 2022 increased by 5.6%.

The analysis of candidates' responses per topic is presented in Section 3.0; where factors that contributed to good, average and weak performance across the topics are highlighted. Finally, the conclusion and recommendations for improvement of performance in the future are provided in Section 4.0 of this report.

2.0 ANALYSIS OF CANDIDATES' PERFORMANCE IN EACH QUESTION

This section presents the analysis of candidates' performance in each question. The National Examination results were based on the score intervals of 80 – 100, 70 – 79, 60 – 69, 50 – 59, 40 – 49, 35 – 39 and 0 – 34 which are defined as excellent, very good, good, average, satisfactory, subsidiary and fail respectively. For the purpose of this report, the performance in each question is described into three categories; good, average or weak under the percentage boundaries of 60 – 100, 35 – 59 and 0 – 34 respectively.

2.1 142/1 ADVANCED MATHEMATICS 1

2.1.1 Question 1: Calculating Device

The question had three parts (a), (b) and (c) which required the candidates to:

(a) Use a non-programmable calculator to find the value of the expression $\frac{\sqrt{\pi^{\cos 60^0}}}{3.14} \times \frac{\left(e^{2.15} + \tan^{-1}(\ln 0.25)\right)}{\sqrt{\pi l^{n\pi}}}$ correct to 3 decimal places.

(b) Use a non-programmable calculator to find the value of $T = \sqrt{\frac{2hp}{g(p-mg)^{\frac{1}{3}}}}$ correctly to nine significant figures given that $h = 3$, $p = 500$, $g = 10$ and $m = 0.25$.

(c) Use statistical functions of a non-programmable calculator and the following frequency distribution table; to find the mean, variance and standard deviation correctly to three decimal places.

Values	250	230	210	190	170	150	130	110	90	70	50	30
Frequency	4	11	5	6	21	40	26	4	8	35	28	12

The analysis of data shows that 13,449 candidates (100%) attempted this question. Among them, 2,716 (20.2%) candidates scored from 0 to 3 marks, while 2,171(16.1%) candidates scored from 3.5 to 5.5 marks and 8,562 (63.7%) candidates scored from 6 to 10 marks. Figure 2 shows the percentage of candidates' performance.

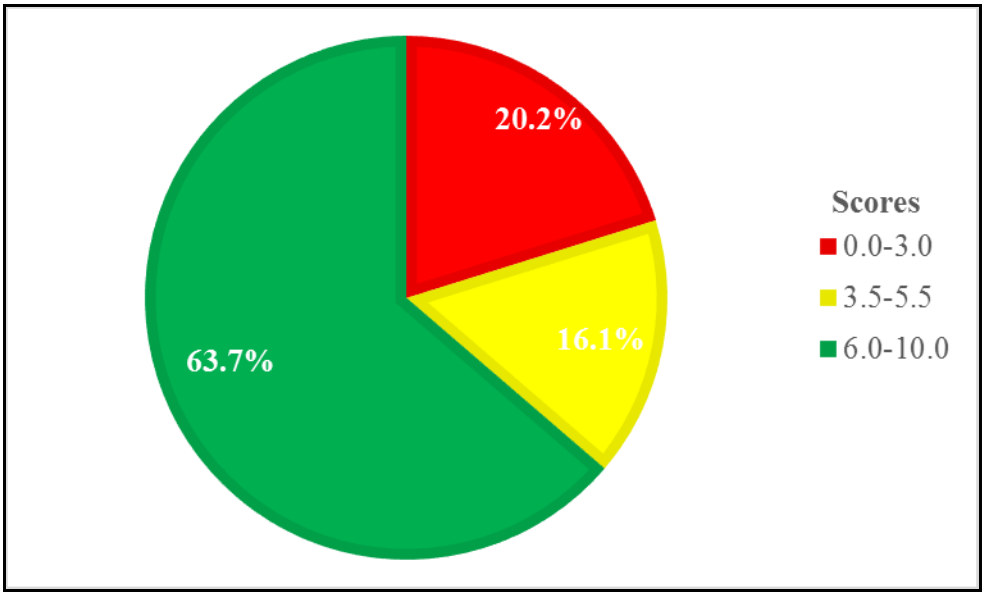
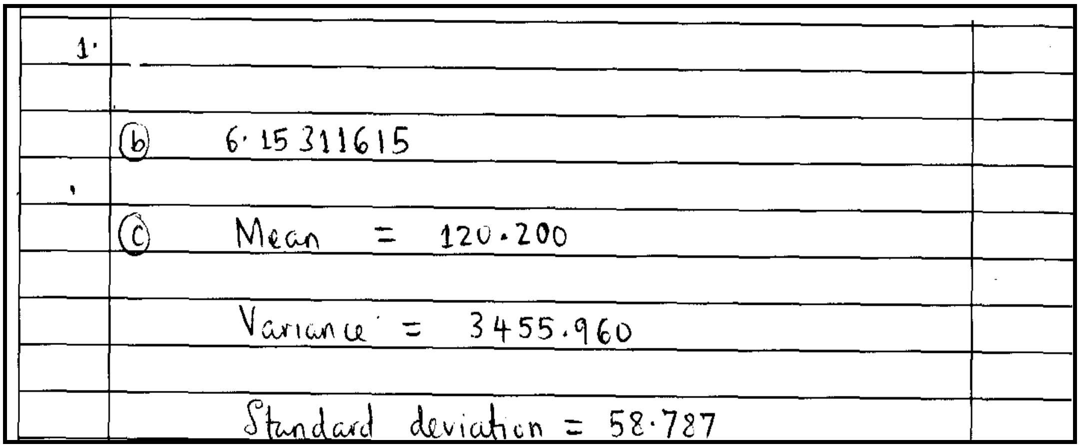


Figure 2: *Candidates' Performance in Question 1*

From Figure 2, it is observed that the candidates' performance in this question was good.

The analysis shows that most candidates had good performance in this question. Thus, the candidates who answered the question correctly were able to use the functional keys of a non-programmable calculator to calculate the required values. For example, candidates were able to get the correct values that are $\text{Mean} = 120.200$, Variance $= 3455.960$ and $\text{Standard deviation} = 58.787$ in three decimal places after plugging the given data into the calculator. Extract 1.1 shows a response from one of the candidates who performed well the question.


1.

(b) 6·15311615

(c) Mean = 120·200

Variance = 3455·960

Standard deviation = 58·787

Extract 1.1: A sample of parts of correct responses to question 1

Extract 1.1 shows that the candidate responded correctly to the question by using the functional keys of a non-programmable calculator to compute the value of T, mean, variance and standard deviation.

Even though most of the candidates showed an outstanding performance in this question, some of them failed to get the correct solution. For example, in part (b), some of them were able to substitute values of $h = 3$, $p = 500$, $g = 10$ and $m = 0.25$ into the given equation of T and finally got $T = 6.15$. Those candidates did not realise that T was supposed to be written in nine significant figures, that is, $T = 6.15311615$. Others tried to rewrite $T^2 = \dfrac{2hp}{g(p-mg)^{\frac{1}{3}}}$ and by setting h, p, g and m, hence got $T^2 = 37.86083836$, concluding that this was the value of *T*. Those candidates did not realize that *T* could be found by taking the square root of 37.86083836, so that *T*=6.15311615.

Also in part (c), it was observed that most of the candidates were able to utilize the correct statistical functions of a non-programmable calculator to find the mean, variance, and standard deviation. But some of them failed to approximate the answers to three decimal places as per instructions. For instance, they got mean of 120.2 instead of 120.200, variance of 345.96 instead of 3455.960 and standard deviation of 58.9356 instead of 58.787. Extract 1.2 is a sample of a candidate who had inadequate competence on what was tested in this question.

1a/		
b/	$T: \sqrt{\dfrac{2(3)(500)}{10(500-0.25(10))^{1/3}}}$	
	$= 1.34500752$	
c/	Mean: 120.2	
	Variance: 58.787	
	Standard deviation: 58.935	

Extract 1.2: A sample of incorrect responses to question 1

In Extract 1.2, the candidate failed to find the correct value of T in part (b) and in part (c), the candidate failed to write the mean into three decimal places while completely got incorrect answers for the variance and standard deviation by using statistical functions of a non-programmable calculator.

2.1.2 Question 2: Hyperbolic Functions

This question comprised three parts (a), (b) and (c). Part (a), required the candidates to obtain the value of x in logarithmic form if $2\cosh 2x + 10\sinh 2x = 5$. While in part (b), the candidates were required to show that $\tanh\theta = -\cos 2\phi$ given that $\theta = \ln(\tan\varphi)$ and in part (c), they were supposed to evaluate the integral $\int_0^1 x\sin 2x dx$ by using the integration by parts technique correctly to 7 decimal places.

The data analysis shows that 13,449 candidates (100%) attempted this question. Out of whom 3,919 (29.1%) candidates scored 0 to 3 marks, while 3,067(22.8%) candidates scored 3.5 to 5.5 marks and 6,463(48.1%) candidates scored 6 to 10 marks. The percentage of candidates' performance is shown Figure 3.

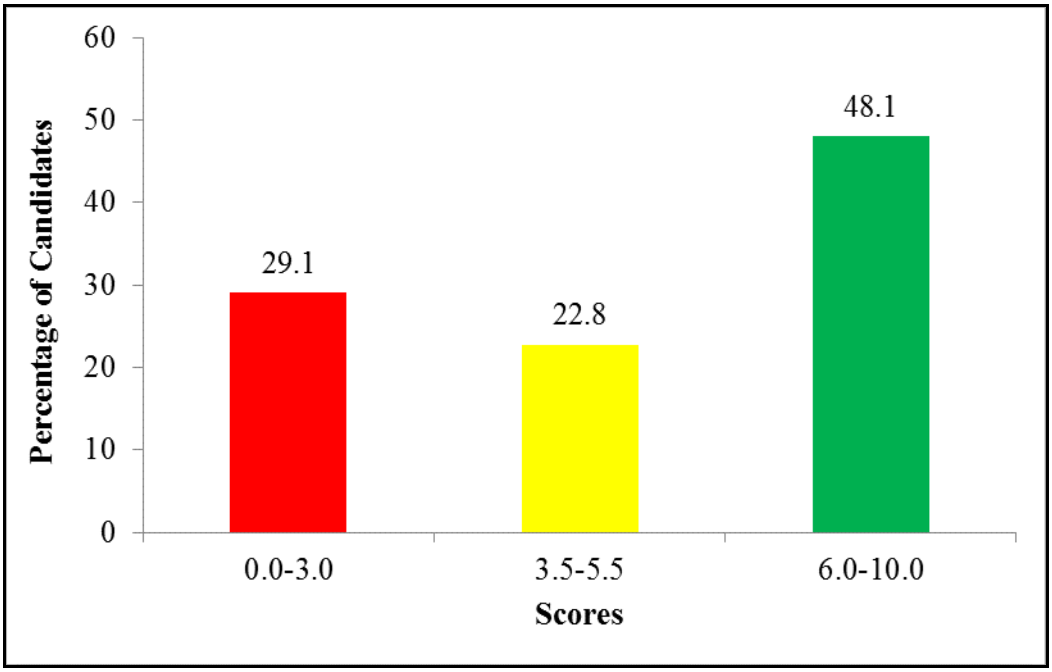


Figure 3: *Candidates' Performance in Question 2*

Figure 3 shows that 70.9 per cent of the candidates scored 3.5 to 10 marks; indicating good performance in this question.

The candidates who performed well in this question, demonstrated a good understanding of the concepts tested. In part (a), those candidates defined correctly $\cosh 2x = \left(\frac{1}{2}\left(e^{2x}+e^{-2x}\right)\right)$ and $\sinh 2x = \left(\frac{1}{2}\left(e^{2x}-e^{-2x}\right)\right)$, then they inserted in the equation $2\cosh 2x + 10\sinh 2x = 5$ and after solving they obtained $x = \frac{1}{2}\ln\left(\frac{4}{3}\right)$.

In part (b), the candidates applied the hyperbolic tangent on both sides of the equation $\theta = \ln\left(\tan\varphi\right)$, to obtain $\tanh\theta = \tanh\ln\tan\phi$ and defined $\tanh\left(\ln\tan\varphi\right)$ as $= \frac{e^{\ln\tan\phi} - e^{-\ln\tan\phi}}{e^{\ln\tan\phi} + e^{-\ln\tan\phi}}$ to get $\tanh\theta = \frac{\tan\phi - \left(\tan\phi\right)^{-1}}{\tan\phi + \left(\tan\phi\right)^{-1}}$, which equals to $\frac{\tan^2\varphi - 1}{\tan^2\varphi + 1} = \frac{\tan^2\varphi - 1}{\sec^2\varphi}$. They simplified this equation further and managed to show that $\tanh\theta = -\cos 2\varphi$.

Candidates who answered part (c) correctly, used the concept of integration by parts. From the formula $\int u\,dv = uv - \int v\,du$, the candidates were able to let $u = x$ and

$dv = \sin 2x dx$. Then, they differentiated u with respect to x that is $\frac{du}{dx} = 1$, hence they obtained $du = dx$ and integrated dv to get $v = -\frac{1}{2}\cos 2x$. Finally, the candidates substituted in the formula above as $\int_0^1 x \sin 2x dx = \left[-\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x \right]_0^1$ to get 0.4353978 as it was required. Extract 2.1 is a sample solution of the correct response from one of the candidates.

02. (a) $2\cosh 2x + 10 \sinh 2x - 5 = 0$.

$2\left(\frac{e^{2x}+e^{-2x}}{2}\right) + 10\left(\frac{e^{2x}-e^{-2x}}{2}\right) - 5 = 0$

$e^{4x}+1 + 5(e^{4x}-1) - 5e^{2x} = 0$.

$e^{4x}+1+5e^{4x} - 5 - 5e^{2x} = 0$.

$6e^{4x} - 5e^{2x} - 4 = 0$.

$6(e^{2x})^2 - 5(e^{2x})' - 4 = 0$.

solving quadratically by general formula.

$e^{2x} = \frac{5 \pm \sqrt{25+96}}{12}$

$e^{2x} = \frac{5 \pm \sqrt{121}}{12}$

$e^{2x} = \frac{5 \pm 11}{12}$

$e^{2x} = \frac{16}{12}$, $e^{2x} = \frac{-6}{12}$.

$2x = \ln \frac{4}{3}$, $2x = \ln(-\frac{1}{2})$ not valid since no logarithm of negative number.

$\therefore x = \frac{1}{2}\ln\left(\frac{4}{3}\right)$

(b) $\theta = \ln(\tan\phi)$.

$2\theta = \ln(\tan^2\phi)$

$e^{2\theta} = \tan^2\phi$ - - - - - (1)

but $\tanh\theta = \frac{\sinh\theta}{\cosh\theta} = \frac{e^{\theta}-e^{-\theta}}{2} \cdot \frac{2}{e^{\theta}+e^{-\theta}}$

$\tanh\theta = \frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}$

$\tanh\theta = \frac{e^{2\theta}-1}{e^{2\theta}+1}$

but from eqn(1) $e^{2\theta} = \tan^2\phi$:

Q2. (b)

$$\tanh\theta = \frac{\tan^2\phi - 1}{\tan^2\phi + 1}$$

but $\tan^2\phi + 1 = \sec^2\phi$.

$$\tanh\theta = \frac{\tan^2\phi - 1}{\sec^2\phi}$$

$$\tanh\theta = \frac{(\tan^2\phi - 1)\cos^2\phi}{(\sec^2\phi)\cos^2\phi}$$

$$\tanh\theta = \cos^2\phi \cdot \tan^2\phi - \cos^2\phi$$

$$\tanh\theta = \sin^2\phi - \cos^2\phi$$

$$\tanh\theta = -\left(\cos^2\phi - \sin^2\phi\right)$$

but $\cos^2\phi - \sin^2\phi = \cos 2\phi$

$\therefore$ $\tanh\theta = -\cos 2\phi$ hence shown.

(c) $\int_0^1 x\sin 2x\, dx$

let $u = x$ $\qquad \int dv = \int \sin 2x\, dx$

$du = dx$ $\qquad v = \frac{-\cos 2x}{2}$

Integral by parts

$$\int u\,dv = uv - \int v\,du.$$

$$\int x\sin 2x\,dx = -\frac{x\cos 2x}{2} - \int \frac{-\cos 2x}{2}\,dx.$$

$$\int x\sin 2x\,dx = \frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}$$

applying limits

$$\int_0^1 x\sin 2x\,dx = \left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^1$$

$$\int_0^1 x\sin 2x\,dx = \frac{\sin 2}{4} - \frac{\cos 2}{2} - \frac{\sin 0}{4} + 0$$

$$\therefore \int_0^1 x\sin 2x\,dx = 0.4353978$$

Extract 2.1: A sample of correct responses to question 2

In Extract 2.1, the candidate was able to solve the value of x in logarithmic form by using the correct definition in part (a). In part (b), the candidate managed to use the definition of hyperbolic tangent to show that $\tanh\theta = -\cos 2\phi$. While in part (c), the candidate correctly used the concept of integration by parts to evaluate the given definite integral.

Despite the good performance in this question, some of the candidates were not able to perform well this question. In part (a), several candidates defined incorrectly the hyperbolic identities. For example, one of the candidates wrote $\sinh x = \frac{e^x + e^{-x}}{2}$, then substituted it in $2\cosh 2x + 10\sinh 2x = 5$ and got $2\left(\frac{e^{2x} + e^{-2x}}{2}\right) + 10\left(\frac{e^x + e^{-x}}{2}\right) = 5$. The equation was simplified to $e^x\left(e^{2x} + 5e^x - 4\right) = 5$. Furthermore, the equation was solved by equating $e^x = 5$ to get $x = \ln 5$ and $x^{2x} + 5e^x - 4 = 0$ to get $x = \ln\left(\frac{-5 \pm \sqrt{41}}{2}\right)$. This was completely wrong because in quadratic equations the product of two factors is equal to zero.

In part (b), some candidates who got incorrect answers defined $\tanh\theta$ using wrong identities as $\tanh\theta = \frac{e^\theta - e^{-\theta}}{2}$ and introduced in the equation $\tanh\theta = -\cos 2\theta$ that is, $\frac{e^\theta - e^{-\theta}}{2} = -\cos 2\phi$. Then solved it to obtain $\theta = \ln\left(\tanh\phi \pm \sqrt{\tanh^2\phi + 1}\right)$. Thereafter, they recalled that $\theta = \ln(\tanh\phi)$ which led them to get $\tan\phi = \tanh\phi \mp \sqrt{\tanh^2 + 1}$. Others defined $\tanh\theta = \frac{e^{2\theta} - 1}{e^{2\theta} + 1}$, and then equated $\tanh\theta$ to some variables. For instance $y = \frac{e^{2\theta} - 1}{e^{2\theta} + 1}$, then simplified it in terms of y as $e^{2\theta} = \frac{y+1}{y-1}$, indicating that $\theta = \frac{1}{2}\ln\left(\frac{y+1}{y-1}\right)$. From this step, they recalled $\theta = \ln(\tan\phi)$ and y as $\tanh\theta$ so that $\ln\tan\varphi = \frac{1}{2}\ln\left(\frac{\tanh\theta + 1}{\tanh\theta - 1}\right)$. Finally they arranged and simplified it to $\tanh^2\phi - \tan^2\phi\tanh^2\theta = 1 + \tanh^2\theta + 2\tanh\theta$. This was also incorrect because it was required to show that $\tanh\theta = -\cos 2\phi$.

In part (c), some candidates separated the integral $\int_0^1 x\sin 2x dx$ into the form $\int_0^1 x dx + \int_0^1 \sin 2x dx$. After working with these separate integrals obtained 2.416146 instead of 0.235 which was required. Those candidates had inadequate knowledge and skills on how to use the formula of integration by parts, $\int u dv = uv - \int v du$ to integrate $\int_0^1 x\sin 2x dx$. Also, in some cases, a few candidates converted $\sin 2x$ into $\sinh 2x$ then stated that $\int_0^1 x\sin 2x dx = \int_0^1 x\sinh 2x dx$. Thereafter, they used the integration by parts technique to obtain 0.9743827 as an equivalent answer to integral $\int_0^1 x\sin 2x dx$. This was contrary to the requirement of the question because it was not directed to change $\sin 2x$ to $\sinh 2x$. Not only that, but also some candidates expressed $\int_0^1 x\sin 2x dx$ as $\int_0^1 2x\sin x\cos x dx$ then they tried to work out by substitution. However, it was difficult to conquer the integral because some variables remained after simplification that is, they considered $u = \sin x,\ du = \cos x dx$ and $\int_0^1 2x\sin x\cos x dx = 2\int \sin x u dx$. Extract 2.2 is a sample of a response by a candidate who failed to demonstrate the intended skills in some parts of the question.

=

(c). $\int_0^1 x\sin 2x\,dx$.

from abam's rule: $\sin x = \sinh x$.

$\sinh 2x = \sin 2x$.

$\int_0^1 x\sin 2x\,dx = \int_0^1 x\sinh 2x\,dx$.

$\int_0^1 x\sinh 2x\,dx$.

let $u = x$.

$du = dx$.

$v = \int \sinh 2x\,dx$

$v = \frac{1}{2}\cosh 2x$.

Recall, $\int u\,dv = uv - \int v\,du$.

$\int u\,dv = \frac{x}{2}\cosh 2x - \int \frac{1}{2}\cosh 2x\,dx$.

$\int u\,dv = \frac{x}{2}\cosh 2x - \frac{1}{2}\int \cosh 2x\,dx$.

$\int u\,dv = \frac{x}{2}\cosh 2x - \frac{1}{2}\left(\frac{1}{2}\sinh 2x\right)$.

$\int u\,dv = \frac{x}{2}\cosh 2x - \frac{1}{4}\sinh 2x$.

$\therefore \int_0^1 x\sinh 2x\,dx = \left[\frac{x}{2}\cosh 2x - \frac{1}{4}\sinh 2x\right]_0^1$

$= 0.9743827 - 0$.

$= 0.9743827$.

$\therefore$ ~~that~~ $\int_0^1 x\sin 2x\,dx = 0.9743827$.

Extract 2.2: A sample of incorrect responses to question 2(c)

In Extract 2.2, part (c) of the question, the candidate incorrectly converted $\sin 2x$ into $\sinh 2x$ then evaluated the given definite integral by using the concept of integration by parts.

2.1.3 Question 3: Linear Programming

This question consisted of two parts, (a) and (b). In part (a), the question stated, "Mr. Safari wants 10, 12 and 12 units of chemicals A, B and C respectively for his garden. A liquid product contains 5 units of A, 2 units of B and 1 unit of C per jar and each jar is sold at 3,000/=. On the other hand, a dry product contains 1 unit of A, 2 units of B and 4 units of C per carton and each carton is sold at 2,000/=. If x and y are the numbers of jars of liquid products and cartons of dry products respectively, formulate a linear programming problem to minimize the cost."

In part (b), the question read "A cement dealer has two depots D1 and D2 holding 180 tons and 250 tons of cement respectively. The customers C1 and C2 have ordered 200 and 150 tons respectively. The transport cost per ton from each depot to each customer are as shown in the following table:"

From	**Customer**	
	C_1	**C_2**
Depot D_1	1,000/=	1,500/=
Depot D_2	2,000/=	1,800/=

Candidates were required to:

i. Find the number of tons of cement which should be delivered to each customer in order to minimize the transport cost?

ii. Find the number of tons of cement that will remain at D_2 after meeting the orders.

The analysis shows that, 66.0 per cent of the candidates who attempted the question scored from 6 to 10 marks, 21.8 per cent scored from 3.5 to 5.5 marks and 12.2 per cent scored from 0 to 3 marks. Generally, the candidates' performance in this question was good as 87.8 per cent of the candidates got not less than 3.5 marks. Figure 4 illustrates the candidates' performance in this question.

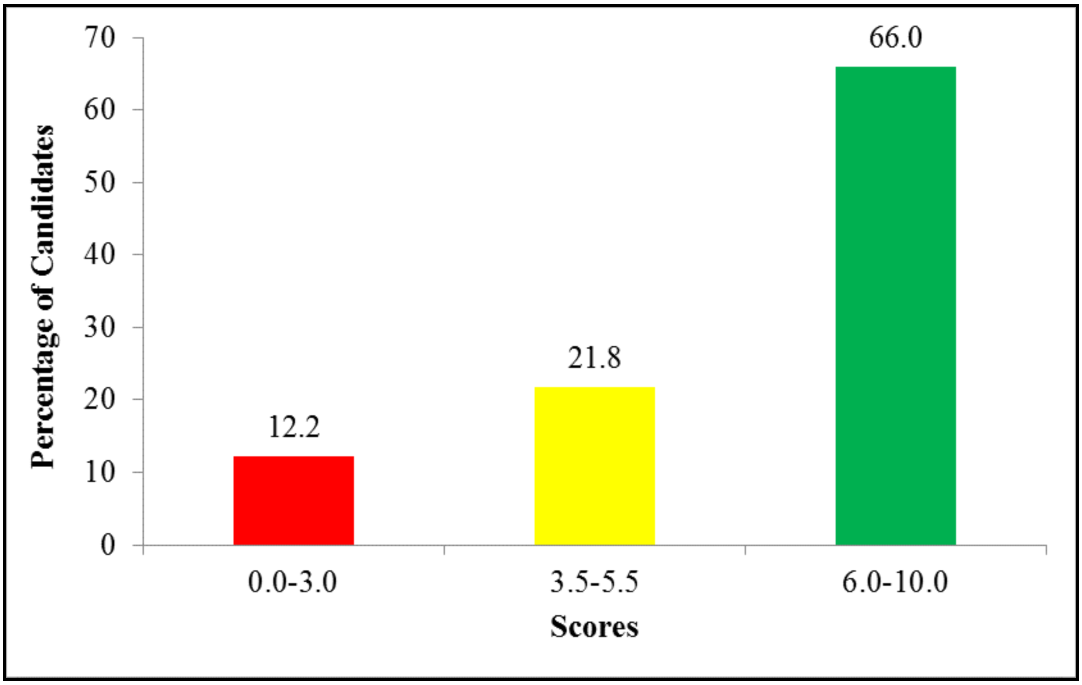


Figure 4: *Candidates' Performance in Question 3*

The analysis of candidates' responses shows that; the majority of the candidates who performed well the question were able to formulate the linear programming model correctly. That is, they first formulated the objective function as: Minimize:

$f(x, y) = 3000x + 2000y$ which is subjected to: $5x + y \geq 10$, $2x + 2y \geq 12$, $x + 4y \geq 12$ and $x, y \geq 0$ as constraints.

Also in part (b) (i), the candidates were able to formulate a mathematical model from the transportation problem as minimize $f(x, y) = 670{,}000 - 1000x - 300y$ subject to $x + y \leq 180$, $x + y \geq 100$, $x \leq 200$, $y \leq 150$ and $x, y \geq 0$. Then represented the formulated constraints on the $xy-$plane, hence got $(0, 100)$, $(0, 150)$, $(30, 150)$, $(180, 0)$ and $(100, 0)$ as the corner points. Thereafter, using the corner points the candidates obtained 180 tons of cement delivered from D_1 to C_1, no cement from D_1 to C_2, 20 tons from D_2 to C_1 and 150 tons of cement from D_2 to C_2. Finally, in part (b) (ii), the candidates were able to get 80 number of tons which remain at D_2 after meeting the orders. Extract 3.1 shows a response from a candidate who answered this question correctly.

03. (a) Decision variables

The number of jars of liquid products be x

The number of cartons of dry products be y.

Objective function

$f(x, y) = 3000x + 2000y$ s/t minimize

Constraints

$5x + y \geq 10$ - - - (i)

$2x + 2y \geq 12$

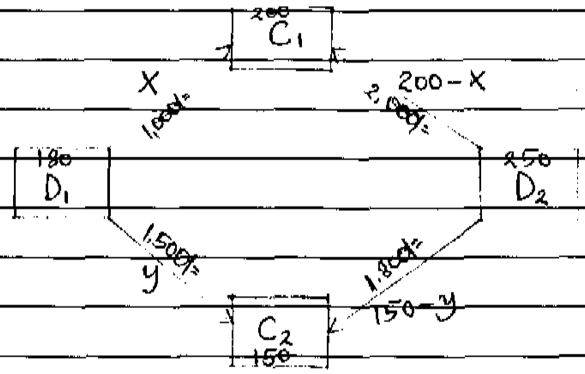
$x + y \geq 6$ - - - - - -(ii)

$x + 4y \geq 12$ - - - - -(iii)

Non-negativity constraints

$x \geq 0$, $y \geq 0$.

Q3 (b)



Decision Variables

Denote number of tons of cement from D_1 to C_1 as x

Denote number of tons of cement from D_1 to C_2 as y.

Objective function

$f(x,y) = 1000x + 1500y + 2000(200-x) + 1800(150-y)$

$f(x,y) = 1000x - 2000x + 1500y - 1800y + 400000 + 270000$

$\underline{f(x,y) = 670000 - 1000x - 300y}$ .. S/t minimize

Constraints

$x \leq 200$, $y \leq 150$

$x + y \geq 100$, $x + y \leq 180$

Non-negative constraints

$x \geq 0$, $y \geq 0$.

Equations

$x + y = 100$

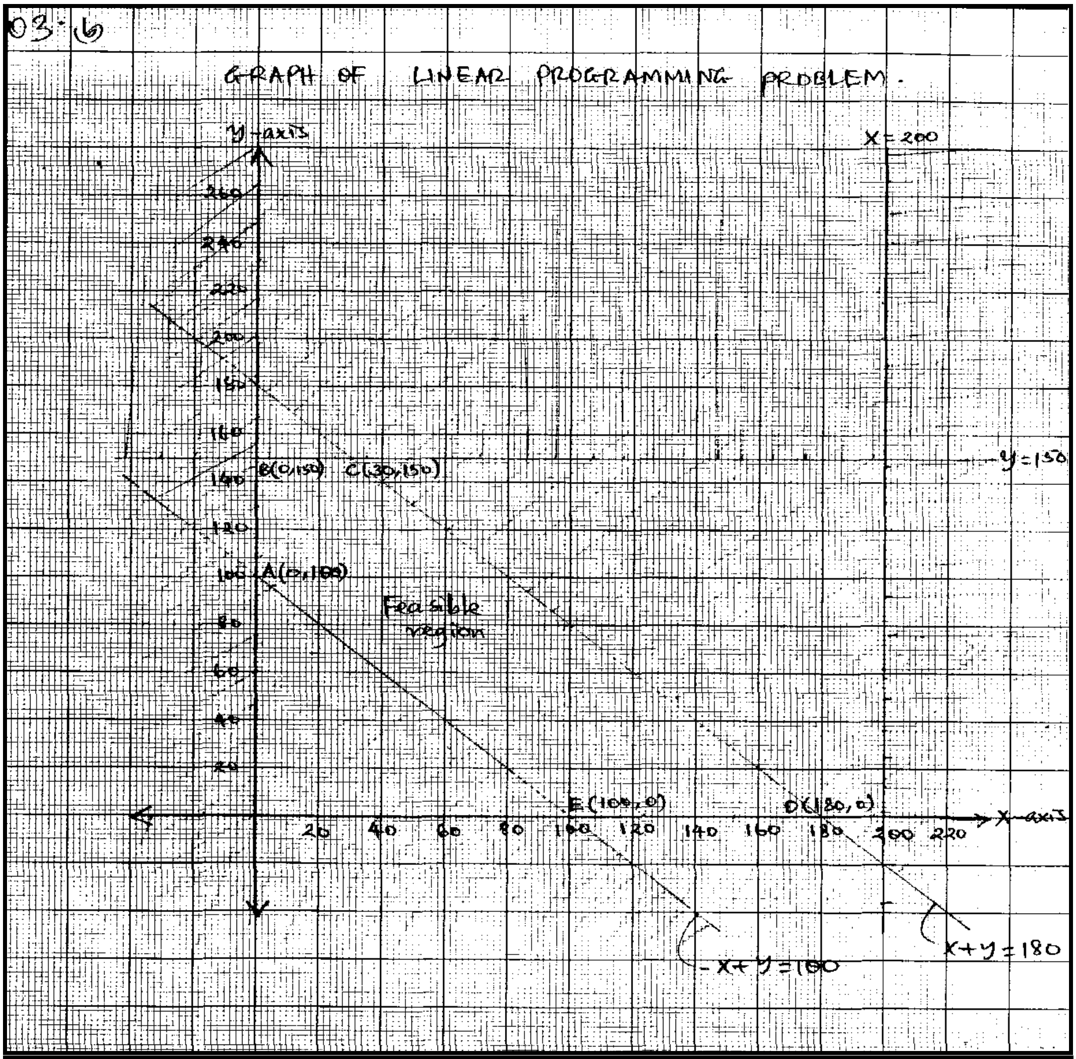
x	y
0	100
100	0

$x + y = 180$

x	y
0	180
180	0

Corner points A(0,100) B(0,150) C(30,150) D(180,0) E(100,0)

03 (b)



03 (b)

C·P	$f(x,y) = 670000 - 1000x - 300y$	Value
A(0,100)	$f(0,100) = 670000 - 1000(0) - 300(100)$	640000
B(0,150)	$f(0,150) = 670000 - 1000(0) - 300(150)$	625000
C(30,150)	$f(30,150) = 670000 - 1000(30) - 300(150)$	595000
D(180,0)	$f(180,0) = 670000 - 1000(180) - 300(0)$	490000
E(100,0)	$f(100,0) = 670000 - 1000(100) - 300(0)$	570 000

Optimal solution is D(180,0).

(i) Inorder to minimize the transport cost, 180 tons of cement should be delivered to customer C_1 from D_1, 20 tons of cement should be delivered to C_1 from D_2 and 150 tons of cement should be delivered to C_2 from D_2.

(ii) 80 tons of cement will remain at D_2.

Extract 3.1: A sample of correct responses to question 3

In Extract 3.1, the candidate was able to formulate a mathematical model from the given word problem in part (a). In part (b) (i), the candidate managed to find the number of tons from the depots D_1 and D_2 to customers C_1 and C_2 and in (b) (ii), the candidate was able to find the number of tons which remained at depot D_2 after meeting the orders.

Despite the fact that most of the candidates attempted well this question, the analysis of the responses revealed that, a few of them faced some challenges. For instance in part (a), those candidates used incorrect inequalities such as $\leq$ instead of $\geq$ as per requirement of the question. For example, some wrote $5x+y\leq 10$, $2x+2y\leq 12$ and $x+4y\leq 12$. Also, they wrote the objective function as $(x,y)=3000x+2000y$ instead of $Minimize: f(x,y)=3000x+2000y$. Others went beyond by taking the summation of all the given data to get the objective function i.e. $f(x,y)=8x+7y+5000$ instead of $f(x,y)=3000x+2000y$. In other circumstances, some candidates established the constraints in the form $5x+2y+z\geq 3000,\ x+2y+4z\geq 2000$. Then they wrote $x\geq 10, y\leq 12$ and $z\leq 12$. Thereafter, they renewed the first two inequalities to $5x+2y\geq 2800$ and $x+2y\geq 1952$. Finally, they stated the objective function as $f(x,y)=x+y+12$. This approach was also wrong because it is contrary to the requirement of the question.

In part (b), a few candidates faced a problem on formulating the linear inequalities. For example the constraints $x+y\leq 200, x+y\geq 280, x\leq 180, y\leq 250, x\leq 0, y\leq 0$ were observed and such candidates ended up with $f(x,y)=-500x+200y+720{,}000$ as the objective function. This kind of mathematical model led them to get the wrong number of tons of cement transported from depots D_1 and D_2 to customers C_1 and C_2. Generally, those candidates had insufficient knowledge and skills on how to formulate mathematical models to solve linear programming problems. Extract 3.2 is a sample of candidate's responses with an incorrect solution to part (a) of this question.

3(a) Let

x to represent chemicals for A ~~required~~

y - to represent chemicals for B required

z to represent chemicals for C required

$A = 10$

$B = 12$

$C = 12$

Inequalities

$5x + 2y + z \geq 3{,}000$

$x + 2y + 4z \geq 2{,}000$

x number of jars and cartons also

y number of jars of liquids and cartons

$5x + 2y + z \geq 3{,}000$

$x + 2y + 4z \geq 2{,}000$

$x \geq 0 \quad y \geq 0$

From

$x \leq 10$

$y \leq 12$

$z \leq 12$

therefore

$5x + 2y + 12 \geq 3000$

$5x + 2y \geq 2{,}988$ - - - (i)

$x + 2y + 4(12) \geq 2{,}000$

$x + 2y \geq 1952$ - - - (ii)

Objective function

$f(x, y) = x + y + 12$

Extract 3.2: A sample of incorrect responses to question 3(a)

In Extract 3.2, the candidate failed to formulate the correct mathematical model to minimize the cost from the given problem.

2.1.4 Question 4: Statistics

The question read "The masses of a sample of new potatoes were measured to the nearest gram and are summarized in the following table:

Mass (g)	10 - 19	20 - 29	30 - 39	40 - 49	50 - 59	60 - 69	70 - 79	80 - 89	90 - 99
Frequency	2	14	21	73	42	13	9	4	2

Candidates were required to determine the following measures of dispersion correct to three decimal places:

(a) first and third quartiles.

(b) semi-inter quartile range.

(c) seventh decile.
(d) 80th percentile.
(e) variance and standard deviation."

Analysis shows that 13,449(100%) candidates attempted the question. Among them, 2,389(17.7%) candidates scored from 0 to 3 marks, 2,847(21.2%) candidates scored from 3.5 to 5.5 marks and 8,216(61.1%) candidates scored from 6 to 10 marks. Figure 5 illustrate the performance of the candidates.

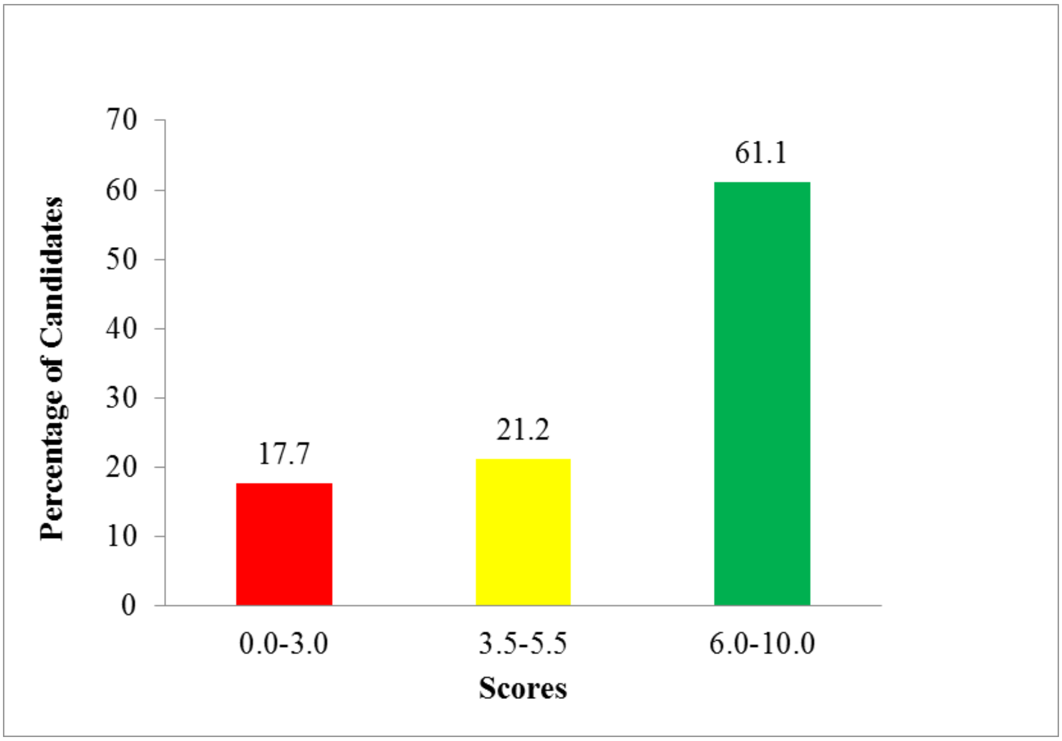


Figure 5: *Candidates' Performance in Question 4*

The performance of candidates to this question was good since a larger number of candidates (82.3%) who attempted the question scored more than 3 marks.

The candidates who performed well in part (a) of this question were able to calculate the lower quartile by using the formula $Q_1 = L + \frac{\left(\frac{N}{4} - f_b\right)}{n_w} \times C$, with $L = 39.5$, $N = 180$, $fb = 37$, $n_w = 73$ and $c = 10$. Hence, they got $Q_1 = 40.596$. On the other hand, the third quartile was successfully calculated by using the formula

$Q_3 = L + \dfrac{\left(\dfrac{3N}{4} - f_b\right)}{n_w} \times C$, then inserted $L = 49.5$, $N = 180$, $fb = 110$, $n_w = 42$ and $c = 10$, finally they got $Q_3 = 55.452$ grams. In part (b), the candidates used the formula $S.I.R = \dfrac{Q_3 - Q_1}{2}$ and substituted $Q_1 = 40.596$ and $Q_3 = 55.452$ to get 7.428 grams as semi-interquartile range.

Similarly in part (c), the candidates used the formula $D_7 = L + \dfrac{\left(\dfrac{7N}{10} - f_b\right)}{n_w} \times C$ to get the seventh decile, while using $L = 49.5$, $N = 180$, $fb = 110$, $n_w = 42$ and $c = 10$ obtained $D_7 = 53.310$ grams as it was required. Not only that, but also in part (d), the candidates used the formula $P_{80} = L + \dfrac{\left(\dfrac{80N}{100} - f_b\right)}{n_w} \times C$ to find the 80th percentile and then inserted $L = 49.5$, $N = 180$, $fb = 110$, $n_w = 42$ and $c = 10$ to obtain $P_{80^{th}} = 57.595$ grams.

Lastly in part (e), the candidates managed to construct a comprehensive frequency distribution table which was a necessary step in solving the question. From the frequency distribution table they obtained $\sum f = 180$, $\sum fu = 66$ and $\sum fu^2 = 384$. Then, they applied the formula $Variance(\mathrm{var}(x)) = c^2\left(\dfrac{\sum fu^2}{\sum f} - \left(\dfrac{\sum fu}{\sum f}\right)^2\right)$ that is $Variance = 10^2\left(\dfrac{384}{180} - \left(\dfrac{66}{180}\right)^2\right)$ to obtain 199.889 as a variance. Lastly, they calculated the standard deviation and got 14.138 by taking the square root of variance. Candidates were keen to write the answers in three decimal places. Extract 4.1, is a sample of a correct response from a candidate who demonstrated a good understanding on the question.

4. (a)

Mass (g)	f	c.f	x	u	fu	u^2	fu^2
10 - 19	2	2	14.5	-3	-6	9	18
20 - 29	14	16	24.5	-2	-28	4	56
30 - 39	21	37	34.5	-1	-21	1	21
40 - 49	73	110	44.5	0	0	0	0
50 - 59	42	152	54.5	1	42	1	42
60 - 69	13	165	64.5	2	26	4	52
70 - 79	9	174	74.5	3	27	9	81
80 - 89	4	178	84.5	4	16	16	64
90 - 99	2	180	94.5	5	10	25	50
					$\Sigma fu = 66$		$\Sigma fu^2 = 384$

(a) 1st quartile

$$Q_1 = L + \left(\frac{N/4 - \Sigma f_n}{f_m}\right)c$$

$N/4 = \frac{180}{4} = 45$ $\quad L = 39.5 \quad c = 10$

$$Q_1 = 39.5 + \left(\frac{45 - 37}{73}\right)10$$

$$\underline{Q_1 = 40.596}$$

Third quartile

$$Q_3 = L + \left(\frac{3/4 N - \Sigma f_n}{f_m}\right)c$$

$3/4 N = \frac{3}{4} \times 180 = 135$

$L = 49.5$

$$Q_3 = 49.5 + \left(\frac{135 - 110}{42}\right)10$$

$$Q_3 = 55.452$$

(b) Semi-interquartile range $= \frac{Q_3 - Q_1}{2}$

$= \frac{55.452 - 40.596}{2}$

Semi-interquartile range $= 7.428$

(c) Seventh decide

$D_7 = L + \left(\frac{\frac{7}{10}N - \sum f_b}{f_m}\right)c$ $\quad \frac{7}{10} \times 180 = 126$

$L = 49.5$

$= 49.5 + \left(\frac{126 - 110}{42}\right)10$

$D_7 = 53.310$

(d) 80^{th} percentile

$P_{80} = L + \left(\frac{\frac{80}{100}N - \sum f_b}{f_m}\right)c$

$\frac{80}{100} \times 180 = 144$ $\quad L = 49.5$

$P_{80} = 49.5 + \left(\frac{144 - 110}{42}\right)10$

$P_{80} = 57.595$

(e) $Var(x) = c^2\left(\frac{\sum fu^2}{N} - \left(\frac{\sum fu}{N}\right)^2\right)$

$= 10^2\left[\frac{384}{180} - \left(\frac{66}{180}\right)^2\right]$

$Var(x) = 199.889$

$S.D = \sqrt{Var(x)}$

$= \sqrt{199.889}$

Standard deviation $= 14.138$

Extract 4.1: A sample of correct responses to question 4

In Extract 4.1, the candidate used the correct formula in calculating the lower quartile, upper quartile, seventh decile, 80^{th} percentile, variance and standard deviation.

On the other hand, there were a few candidates who performed poorly in this question. In part (a), some candidates used wrong formulae in calculating the first and third quartiles. For instance, to find Q_1, they used the formula $\bar{x} = L + \frac{\left(\frac{N}{2} - \sum fb\right)}{n_w} \times i$, with $L = 49.5$, $\sum fb = 10$, $n_w = 4$ and $i = 10$ hence obtaining $\bar{x} = 45$ and treated this as Q_1 instead of $Q_1 = 40.596$ grams. Likewise, for Q_3, they used $\bar{x} = L + \frac{\left(\frac{N}{2} - \sum fb\right)}{n_w} \times i$, with $L = 39.5$, $\sum fb = 37$, $n_w = 73$ and $i = 10$ to obtain $\bar{x} = 41.998$ as the third quartile instead of 55.452 grams. Another candidate used the formula $Q_1 = \frac{1}{4} N^{th}$ and obtained $Q_1 = 37$ and $Q_3 = \frac{3}{4} N^{th}$ hence obtained $Q_3 = 110$.

Also in part (b), a few candidates used the improper formula $SIR = \frac{Q_3 + Q_1}{2}$ to calculate the semi-interquartile range, leading them to get 48.024 instead of 7.428 grams. In part (c), some candidates calculated the seventh decile by using inappropriate formula, for example one candidate used the formula $x = L + \frac{\left(\frac{N}{2} - \sum fb\right)}{n_w} \times i$, with $L = 49.5$, $\frac{N}{2} = 126$, $\sum fb = 110$, $n_w = 42$ and $i = 10$ hence, he/she obtained $x = 55.937$ instead of 53.310 grams as a seventh decile. Those candidates did not realise that the seventh decile is expressed as D_7 and the formula used was that for calculating the median. In part (d), some candidates used wrong formulae to calculate the 80^{th} percentile, such as $P_{80} = \frac{80}{100} N^{th}$ hence, obtained $P_{80} = 110$.

In part (e), some candidates faced a challenge on rounding off the answers for variance and standard deviation into three decimal places. The common answers which were observed from the candidates were $\text{Var}(x) = 199.917, 199.9, 199.89$ and the standard deviation of 14.139, 14.1386, 13.138. Instead of the variance of 199.889 and standard deviation of 14.138. Others utilized unsuitable formulae to calculate the variance and standard deviation. There are a few candidates who used $Var(x) = \frac{\sum x^2}{N} - \left(\frac{\sum x}{N} \right)^2$ with $\sum x^2 = 327732$ and $\sum x = 490.5$ then, they obtained incorrect value of $Var(x) = 174.420$ and $SD(x) = 13.207$. Likewise, some candidates used the formula $Var(x) = A + c\frac{\sum fx}{\sum f}$ and substituted $A = 44.5$, $c = 10$, $\sum fx = 8670$, $\sum f = 180$ hence they got incorrect answer of $Var(x) = 526{,}167$ and $SD(x) = 22.938$. Those candidates lacked knowledge on statistical formulae. Not only that but also the formula $Var(x) = c^2\left(\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f} \right)^2 \right)$ was used. Then substituted $\sum fx^2 = 8670$, $\sum fx = 180$, $c = 10$ to obtain $Var(x) = 150{,}092.173$ and $SD(x) = 387.417$. Those candidates could not realize that c in the formula was suitable when computing variance by using the coding method. Extract 4.2 shows a response from a candidate who performed poorly in this question in parts (a) and (c).

41 (a) 1st quartile = 41.998.

3rd quartile

$X = L + \left(\frac{N/2 - sfb}{nw} \right) c$

$X = 49.5 + \left(\frac{135/2 - 110}{42} \right) 10$

$X = 49.5 + \left(\frac{25}{42} \right) 10$

$X = 49.5 + 5.583$

$X = 55.083$

$$\bar{X} = L + \left(\frac{N/2 - \Sigma fb}{nw}\right) i$$

$$X = 39.5 + \left(\frac{45 - 37}{73}\right) 10$$

$$X = 39.5 + \left(\frac{8}{73}\right) 10$$

$$X = 39.5 + (0.10958)^{10}$$

$$X = 39.5 + 2.498$$

$$X = 41.998$$

4. (c) from

$$X = L + \left(\frac{N/2 - \Sigma fb}{nw}\right) i$$

$$X = 49.5 + \left(\frac{126 - 110}{42}\right) 10$$

$$X = 49.5 + \left(\frac{16}{42}\right) 10$$

$$X = 49.5 + (0.38095)^{10}$$

$$X = 49.5 + 6.4373$$

7^{th} decile $= 55.937$

Extract 4.1: A sample of incorrect responses to question 4

In Extract 4.1, the candidate did not recall the correct formulae for calculating quartiles and deciles to attempt the question.

2.1.5 Question 5: Sets

This question comprised three parts (a), (b) and (c). Part (a), required the candidates to evaluate $A-B$ in the same set notation if sets A and B are defined by $A=\{x\in\Re:-1\leq x\leq 2\}$ and $B=\{x\in\Re:2\leq x<5\}$. The second part of this question was part (b), which required the candidates to simplify $[(A-B)\cup(A\cap B)]-[A\cup(A\cap B)]$ by using laws of algebra of sets. In the last part of the question, part (c), the candidates were given a word problem which read: In a class of 17 girls and 15 boys, 22 play handball, 16 play basketball, 12 of the boys play handball, 11 of the boys play basketball, 10 of the boys play both basketball and handball, 3 of the girls play neither of the two games. Then, the candidates were required to use Venn diagram to determine:

(i) The number of girls who play both games.
(ii) The number of participants who play at least one game.

The analysis of data shows that a total of 12,290(91.4%) candidates scored from 3.5 to 10 marks out of 13,449(100%) candidates who attempted the question. While 1,159(8.6%) candidates scored from 0 to 3 marks. Generally, the performance of candidates to this question was good. Figure 6 provides the candidates' performance to this question.

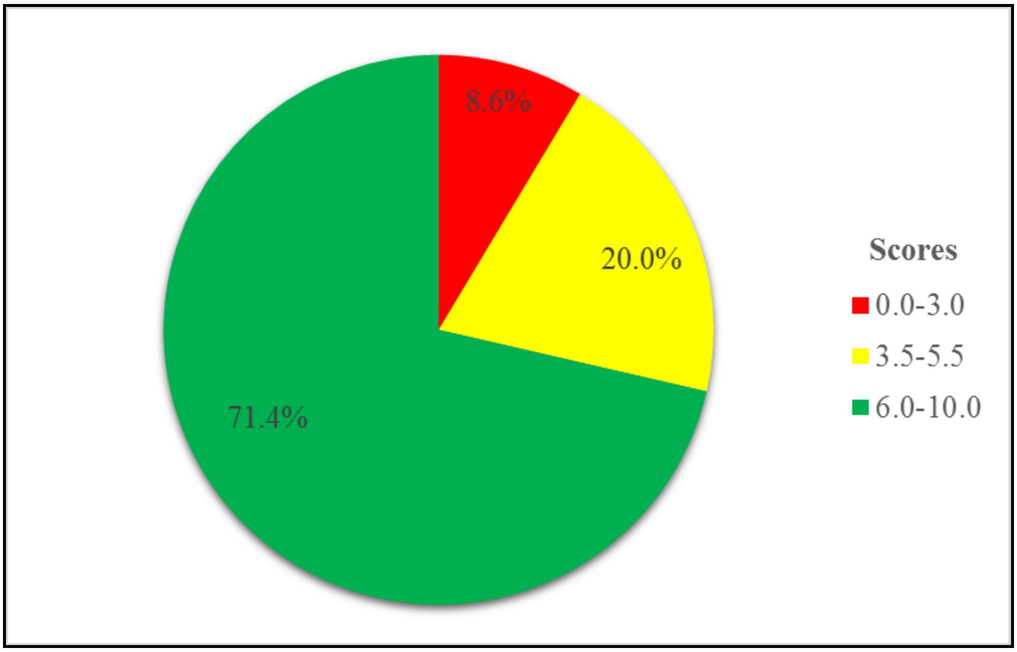
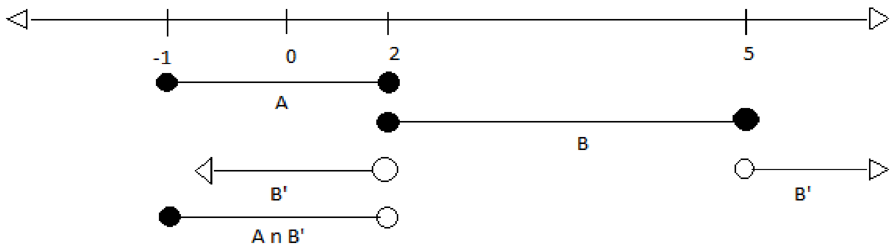


Figure 6: *Candidates' Performance in Question 5*

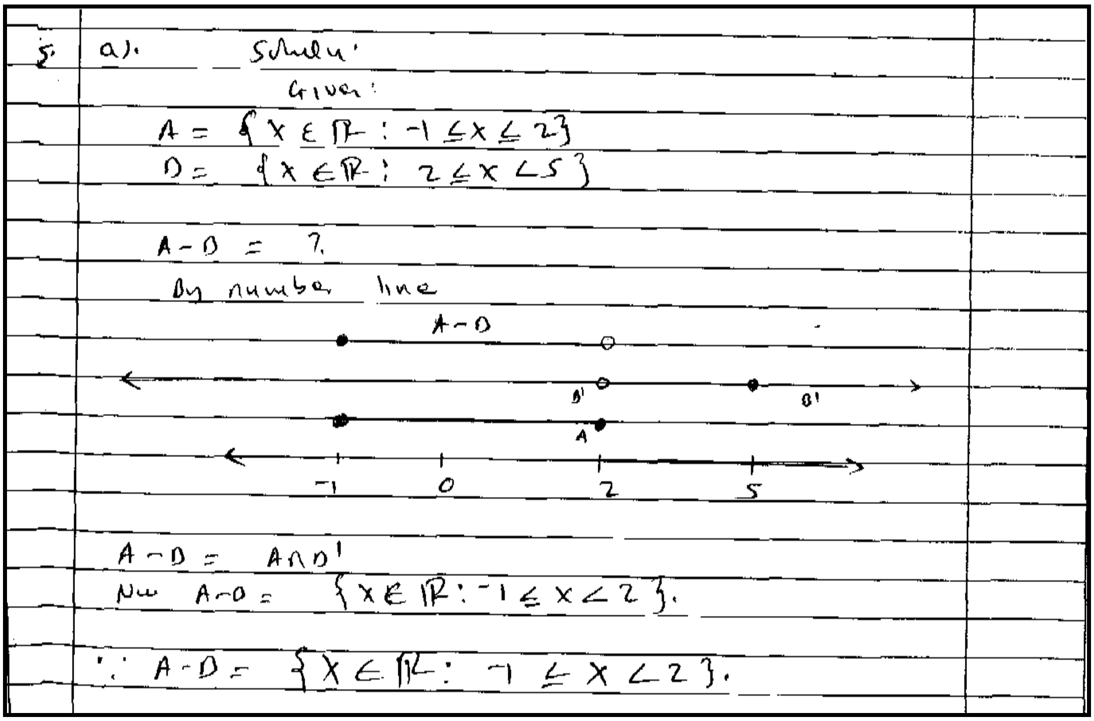
A large number of candidates who answered the question correctly were able to define the set difference law which is $A - B = A \cap B'$ and applied to the given set $B = \{x \in \Re : 2 \le x < 5\}$ to obtain $B' = \{x \in \Re : x < 2 \cup x \ge 5\}$. Then, they used the law of intersection to obtain $A \cap B' = \{x : -1 \le x < 2\}$ which was the required response in part (a). Also, others used the approach of the number line as follows:



(01

Then from the number line they obtained $A \cap B' = \{x \in \Re : -1 \le x < 2\}$.

In part (b), the candidates were able to apply the laws of algebra of sets to simplify $[(A - B) \cup (A \cap B)] - [A \cup (A \cap B)]$ to obtain ϕ. In part (c), the candidates managed to use Venn diagram to determine the number of girls who played both games which is 1, and the number of participants who played at least one game which was 27. Extract 5.1 shows the work of a candidate who answered this question correctly.



5. a). Solution:

Given:

$A = \{x \in \mathbb{R} : -1 \le x \le 2\}$

$B = \{x \in \mathbb{R} : 2 \le x < 5\}$

$A - B = ?$

By number line

$A - B = A \cap B'$

Now $A - B = \{x \in \mathbb{R} : -1 \le x < 2\}$.

$\therefore A - B = \{x \in \mathbb{R} : -1 \le x < 2\}$.

5. b) Soln

Given:

[(A − B) ∪ (A∩B)] − [A ∪ (A∩B)] given

≡ [(A∩B′) ∪ (A∩B)] ∩ [A ∪ (A∩B)]′ definition

≡ [A∩(B′∪B)] ∩ [A∪(A∩B)]′ distribution law

≡ [A∩U] ∩ [A∪(A∩B)]′ complement law

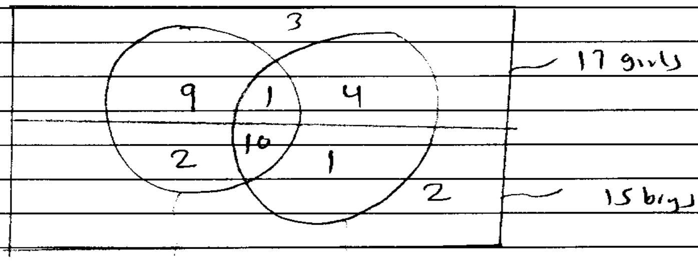
≡ [A ∩ [A∪(A∩B)]′ identity law

≡ A ∩ (A)′ absorption law

≡ A ∩ A′ demorgan law

≡ ϕ complement law

5. c) Soln.



22 handball

16 basketball

i) number of girls who play both games is 1.

ii) number of participant who play at least one game is 27

Extract 5.1: A sample of correct responses to question 5

In Extract 5.1, part (a), the candidate was able to find $A-B$ from the given sets by using the number line. Whereas in part (b), the candidate applied laws of algebra of sets to simplify $[(A-B)\cup(A\cap B)]-[A\cup(A\cap B)]$. Likewise, in part (c), the candidate managed to use the Venn diagram to determine the number of girls who played both games and of the participants who played at least one game.

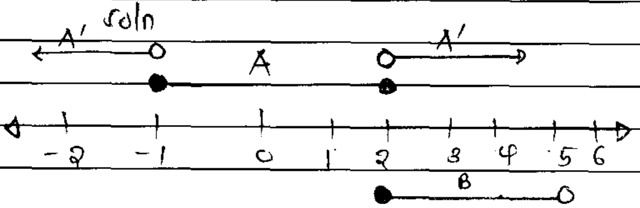
Although majority of the candidates performed this question well, there were a few who got this question wrongly. Analysis shows that in attempting this question, the candidates encountered some challenges. In part (a), some candidates defined wrongly $A-B$ as $A-B=A\cup B'$ and others as $\boldsymbol{A-B=A'\cap B}$. Also the candidates failed to follow instructions in spite of being instructed to use the set builder notation, they used listing method as $A-B=\{-1,0,1\}$ which was incorrect. There were also candidates who had insufficient knowledge on the types of numbers which led them to use interchangeably integers and real numbers.

In part (b), they failed to use the proper algebra laws of set to simplify the given statement. For instance, some candidates defined wrongly the set difference property. In part (c), the number of candidates failed to map the given information on Venn diagram, as a result they got wrong answers in part (c) (i) and (ii). Extract 5.2 shows a sample of responses from the candidate who had no competence in answering this question properly.

a/ Given $A = (-1 \leq X \leq 2)$; $B = (2 \leq X < 5)$

Required to find $A - B$

soln



Thus $(A - B) = A' \cap B$

$= 2 < X < 5$

5b Solution

Given

$[A-B) \cup (A \cap B)] - [A \cup (A \cap B)]$

$[(A-B) \cup (A \cap B)] - [A \cup (A \cap B)]$ --- given

$[(A \cap B') \cup (A \cap B)] \cap (A' \cap (A \cup B)']$ --- Complement law

$[(B' \cap A) \cup (A \cap B)] \cap (A' \cap (B \cup A)']$ -- associative law

$(A \cup A) \cap (B' \cap B) \cap (A' \cap A) \cup B$ --- Distributive law

$(A \cap \emptyset) \cap (\emptyset \cup B)$ --- Identity law

$(A \cap B)$ ------ Identity law

Extract 5.2: A sample of incorrect responses to question 5

In Extract 5.2, the candidate failed to define $A-B$ as $A \cap B'$ in part (a). Also, in part (b), the candidate used wrong laws of algebra of sets to simplify $[(A-B) \cup (A \cap B)] - [A \cup (A \cap B)]$.

2.1.6 Question 6: Functions

The question had two parts; (a) and (b). In part (a), the candidates were given that $f(x)=2^x$ and $g(x)=\log_2 x$, and were required to (i) find the domain and range of $f(x)$ and $g(x)$, in (ii) to draw the graphs of $f(x)$ and $g(x)$ on the same axes and to comment on the resulting graphs. Part (b), the candidates were given that $f(x)=\frac{2x-x^2}{x^2-2x-3}$ and then asked to :

(i) write the values of x for which $f(x)=0$ and the values of x for which $f(x)>0$.

(ii) show that $f(x)\to -1$ as $x\to\pm\infty$.

(iii) sketch the graph of $f(x)$ by showing particularly where the curve crosses the x-axis and how it approaches its asymptotes.

This question was attempted by 13,449 (100%) candidates. The summary of candidates' performance in this question is presented in Figure 7.

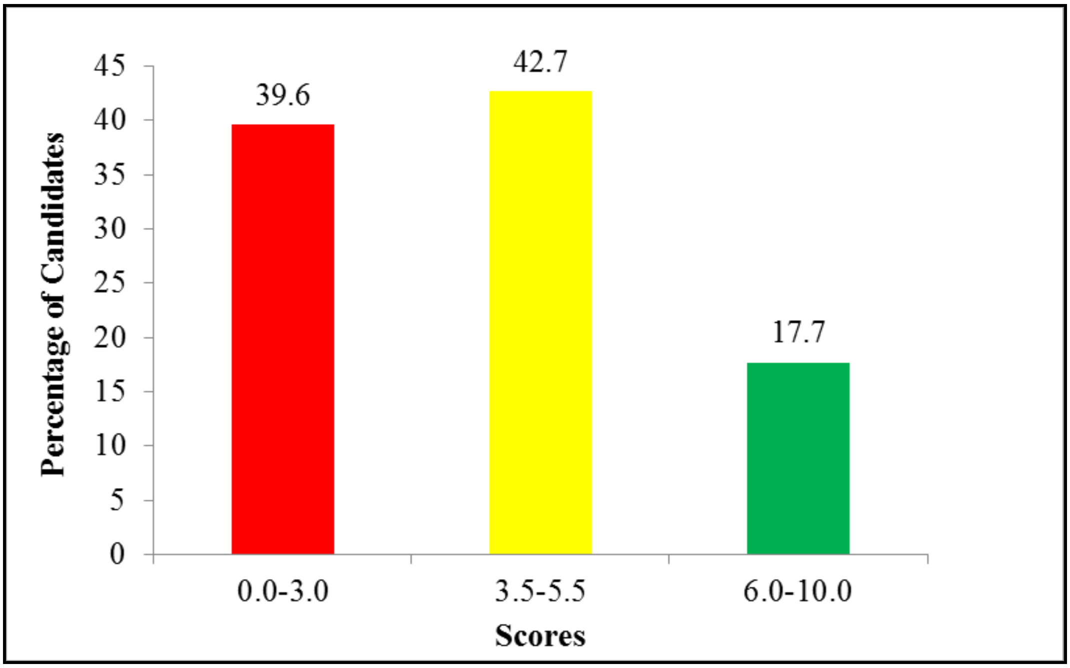


Figure 7: *Candidates' Performance in Question 6*

Analysis shows that among 13,449 candidates who attempted the question, 5,328(39.6%) candidates scored 0 to 3 marks. While 8,121(60.4%) candidates scored

3.5 to 10 marks. Therefore, the performance of the candidates in this question was good.

Those candidates who responded correctly to the question in part (a) (i), were able to obtain domain of $f(x)=2^x$ as $\{x : x \in \Re\}$. They also determined the range of $f(x)$ by making *x* the subject from $y=2^x$ to $x=\log_2 y$ and investigated the resulting function hence they obtained its range $=\{y : y>0\}$. To obtain the domain of $g(x)=\log_2 x$ the candidate applied the concept that, since there is no logarithm of zero or negative number, then domain $=\{x : x>0\}$. Then, they obtained the range of $g(x)$ by making *x* the subject from $y=\log_2 x$ to $x=2^y$ and from this they got the range $=\{y : y \in \Re\}$. In part (a) (ii), they managed to plot both graphs of $f(x)$ and $g(x)$ on the same $xy-$plane and from the graphs commented that $g(x)$ is the image of $f(x)$. Others commented that $g(x)$ is the inverse of $f(x)$ and vice versa.

In part (b) (i), the candidates solved correctly the value of *x* for which $f(x)=0$ and obtained either $x=0$ or $x=2$. Also, for $f(x)>0$, the candidates got the solution set as $\{x \in \Re : -1<x<0 \quad \text{or } 2<x<3\}$ after solving the inequality $\dfrac{2x-x^2}{x^2-2x-3}>0$ and got the critical values as $x=0$ or $x=2$, $x=-1$ and $x=3$.Thereafter, they tested the solution using table and others by using the number line. For part (b) (ii), the candidates obtained $f(x)=-1$ after applying the limit to $f(x)\to -1$ as $x \to \pm\infty$. That is, $f(x)=\dfrac{2x-x^2}{x^2-2x-3}=\dfrac{\frac{2}{x}-1}{1-\frac{2}{x}-\frac{3}{x^2}}$, then as $x \to \pm\infty$ $f(x)=\dfrac{0-1}{1-0-0}=1$ hence shown that $f(x)\to -1$ as $x \to \pm\infty$. In part (b) (iii), the candidates obtained the turning point $\left(1, -\frac{1}{4}\right)$ after finding $f'(x)$ by setting it equal to zero and solving for *x* then substituting it in $f(x)$ to obtain the value of *y*. Not only that, but also the candidates managed to compute both horizontal asymptote and vertical asymptote and used it as a reference to sketch the graph. Extract 6.1 is a sample of responses of a candidate who answered this question correctly.

6. (a) Given

$f(x) = 2^x$ and $g(x) = \log_2 x$

(i) Domain and range for $f(x)$ and $g(x)$

Soln

for $f(x)$

$f(x) = 2^x$.

let $f(x) = y$

$y = 2^x$

for domain make y the subject

$y = 2^x$

On substituting any real number of x it gives real number of y

then

Domain $= \{x : x \in \mathbb{R}\}$.

6(a)(i) for range make x the subject

from

$y = 2^x$

$\ln y = x \ln 2$

$x = \frac{\ln y}{\ln 2}$

$x = \frac{\ln y}{\ln 2}$

Since there is no logarithm of negative number

then $y > 0$

$\therefore$ Range $= \{y : y > 0\}$

for $g(x) = \log_2 x$

Domain

Make ~~y~~ the subject

let $g(x) = y$.

$y = \log_2 x$

Since there is no logarithm of negative number

then $x > 0$

$\therefore$ Domain $= \{x : x > 0\}$.

Range.

Make x the subject.

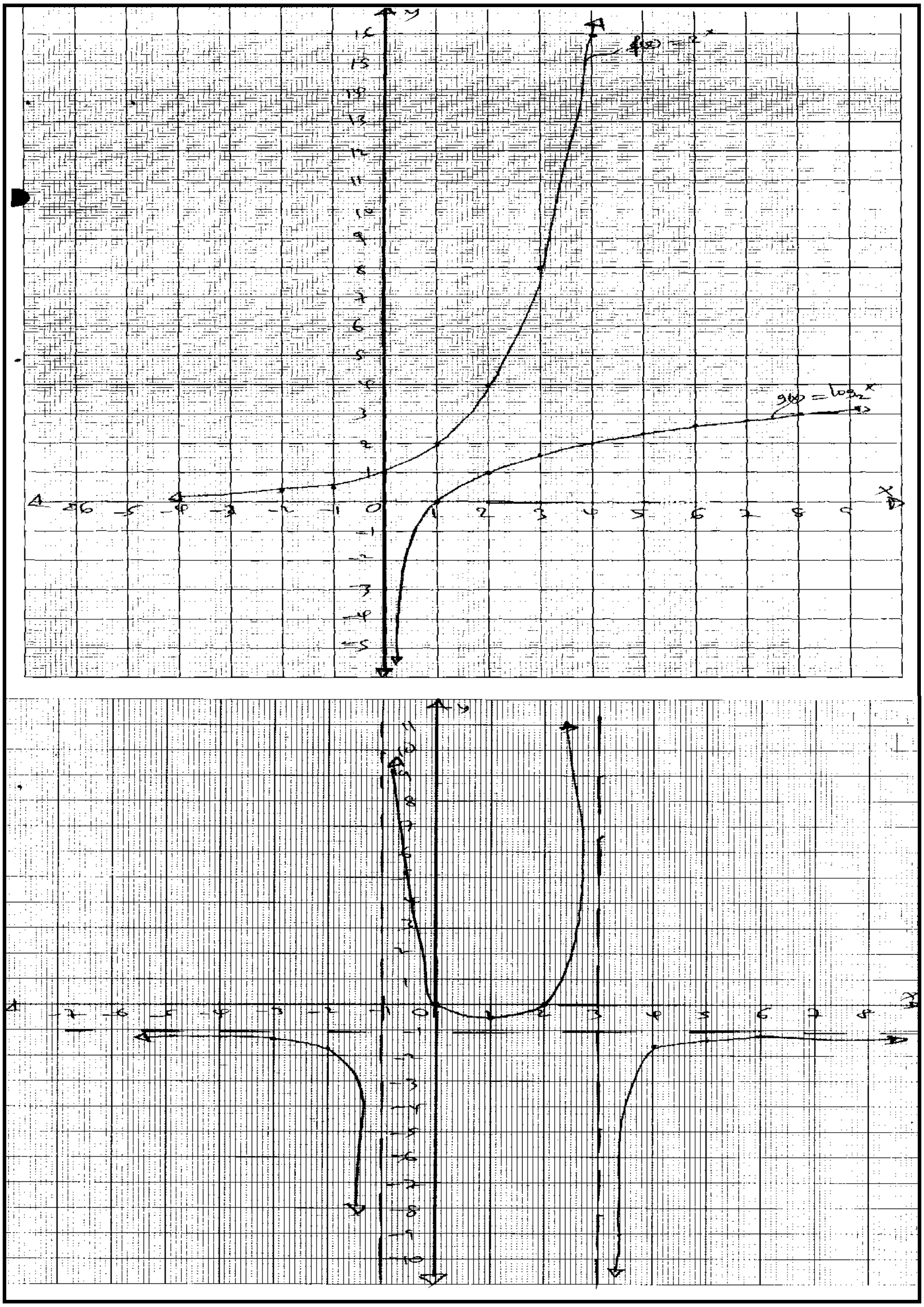
$y = \log_2 x$.

$2^y = x$.

$x = 2^y$.

On substituting real number of y it gives real number of x then;

a(i) Range $= \{y : y \in \mathbb{R}\}$.



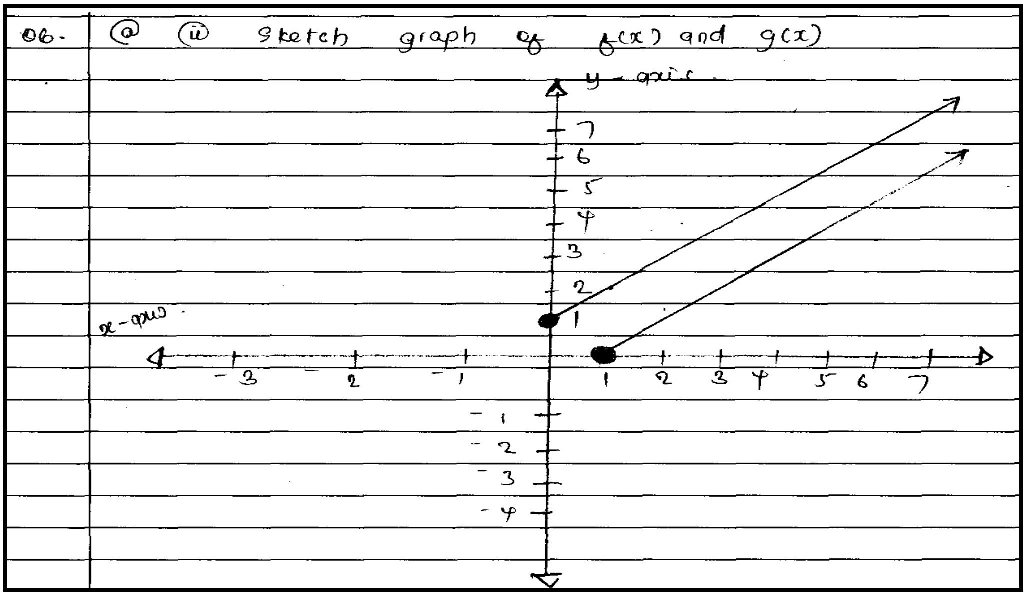
Extract 6.1: A sample of correct responses to question 6

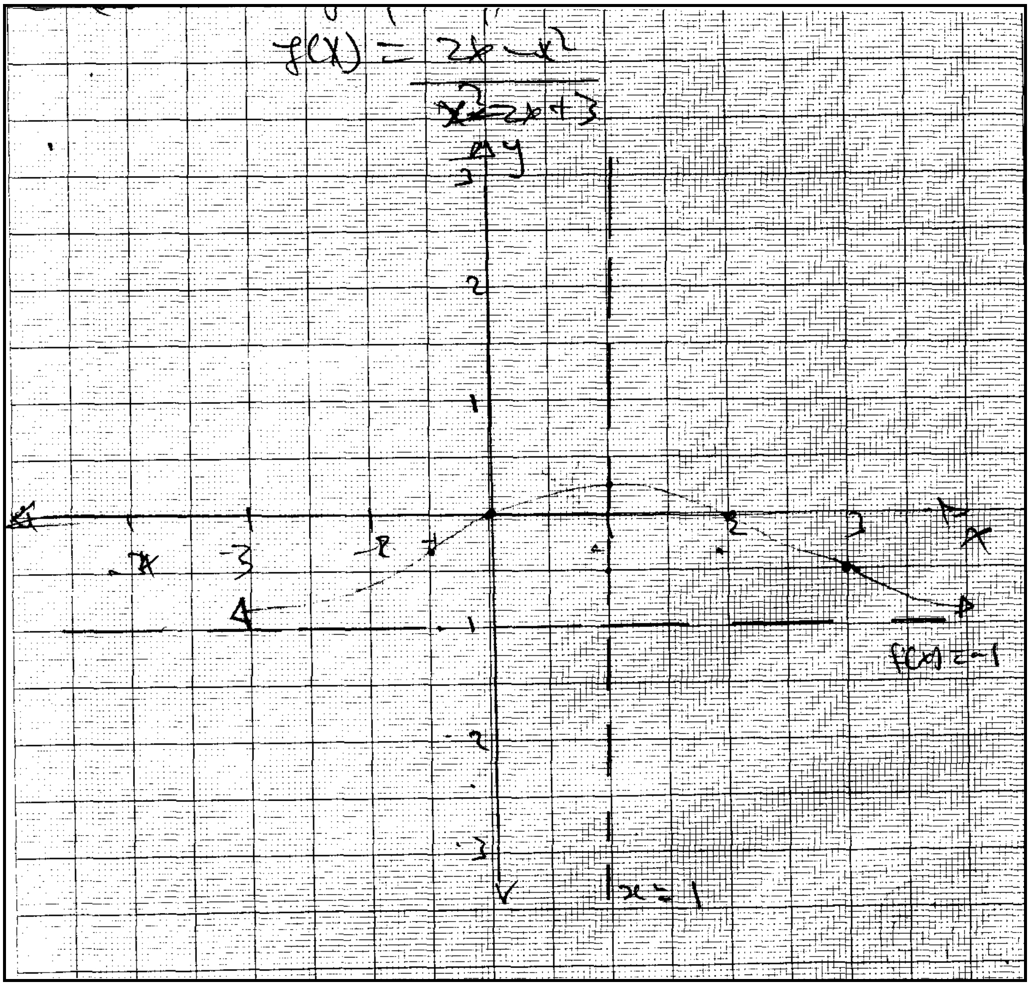
Extract 6.1, the candidate determined correctly the domain and range of exponential and logarithmic functions and managed to sketch its graph accurately in the same xy-

plane in part (a). While in part (b), the candidate sketched well the graph of the given rational function $f(x)$ as instructed.

Despite the good performance in this question, there were a few candidates who did not perform well due to some challenges. In part (a), they sketched the graph instead of drawing it. In this case, the candidates were required to prepare a table of values for both $f(x)$ and $g(x)$ then plot them on the $xy-$plane and join it with free hand. Also, they seem to lack enough knowledge of exponential function such as $f(x)=2^x$ whose graph must cross $y-$axis at (0, 1) and its asymptote $x=0$ and for logarithmic function such as $g(x)=\log_2 x$, its graph always crosses $x-$axis at (1, 0) and $y=0$ is its asymptote. Other candidates did not consider the asymptotes and the intercepts due to lack of enough knowledge, hence they sketched straight lines instead of exponential and logarithmic curves.

Minority of the candidates who answered part (b) (iii) wrongly did not consider the rule of sketching the graph of rational functions. They computed well the vertical and horizontal asymptotes but sketched the lines by using full lines and did not take into consideration that the curve cannot cross the vertical asymptote. Extract 6.2 illustrates incorrect response from one the candidates who attempted this question.





Extract 6.2: A sample of incorrect responses to question 6

In Extract 6.2, the candidate sketched wrongly the graphs of exponential and logarithmic functions in part (a). Likewise, in part (b), the candidate failed to sketch the graph of rational functions as instructed.

2.1.7 Question 7: Numerical Methods

This question consisted of parts (a), (b) and (c). In part (a), the candidates were asked to verify that $x^2 - 2x - 1 = 0$ has a root in the range $2 \le x \le 3$. Part (b) of the question required the candidates to use Secant method with four iterations to obtain an approximation for the root of the equation in (a) correct to three decimal places. Part (c) is divided into three parts (i), (ii) and (iii). In part (c) (i), the candidates were required to use Trapezoidal rule with 5 strips, to obtain an approximation of $\int_0^1 x^2 e^x dx$ correct to four significant figures. In part (c) (ii), the candidate were asked to use the integration by parts technique to evaluate the exact value of the definite integral $\int_0^1 x^2 e^x dx$ correct to four significant figures. And in part (c) (iii) the candidate where instructed to use the results obtained in part (c) (i) and (ii) to find an absolute error in the value of $\int_0^1 x^2 e^x dx$.

Out of 13,449 (100%) candidates who attempted this question, 4,733(35.2%) candidates scored 6 to 10 marks, 4,265(31.7%) candidates scored 3.5 to 5.5 marks while 4,451(33.1%) candidates scored 0 to 3 marks. Figure 8 presents the summary of candidates' performance in this question.

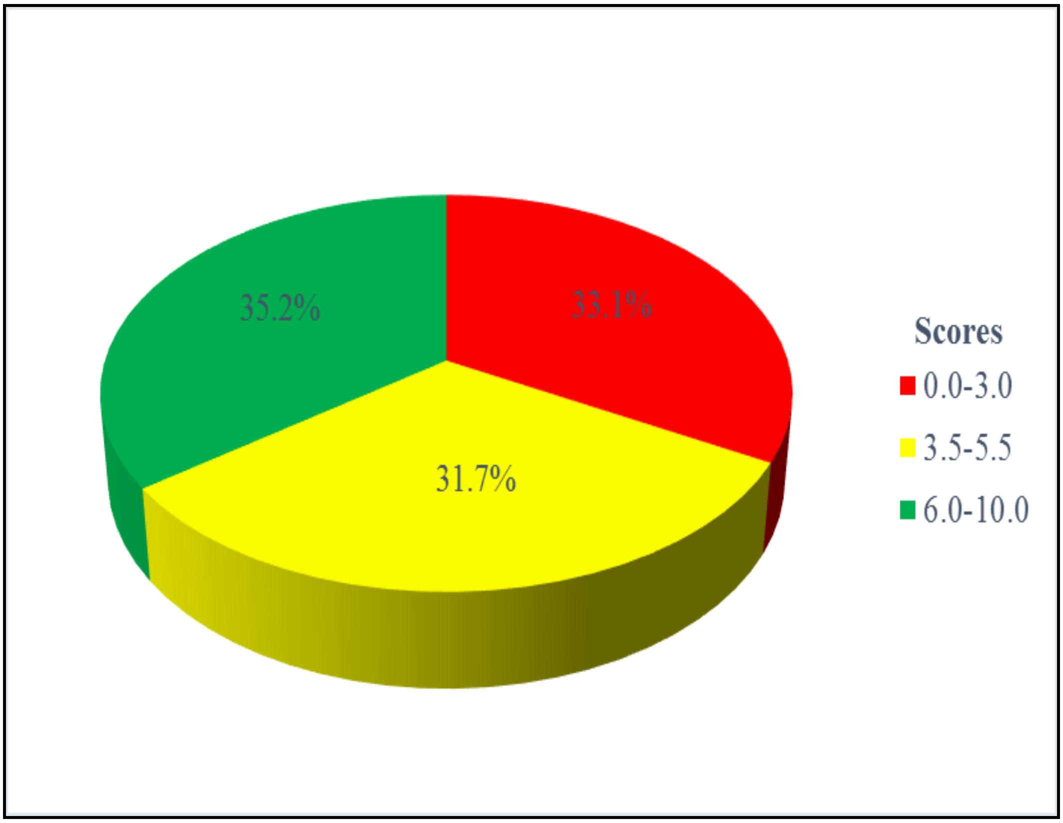


Figure 8: *Candidates' Performance in Question 7*

From Figure 8, it is observed that the candidates' performance in this question was good since 66.9 per cent of the candidates scored within the range of 3.5 to 10 marks.

The majority of the candidates who scored high marks had knowledge of numerical integration. In part (a), the candidates applied $f(x_1).f(x_2)<0$ to check if the given function had the root in the given range. They calculated the value of $f(2)$ and $f(3)$

which were -1 and 2 respectively. Then, since $f(x_1).f(x_2) = -2 < 0$ hence they concluded that there exists a root between $x = 2$ and $x = 3$.

In part (b), the candidates were familiar with Secant formula and applied it after recalling it as $x_{n+2} = x_n - \frac{(x_{n+1} - x_n)}{f(x_{n+1}) - f(x_n)} \times f(x_n)$ and carried out four iterations. In the first iteration when $n = 1$, the candidates used $x_1 = 2$ and $x_2 = 3$ to obtain $x_3 = 2.33333$, then they performed the next iteration until the fourth iteration was reached and got the root 2.414 to three decimal places. In part (c) (i), they were able to find the value of h and obtained $h = 0.2$ by using the formula $h = \frac{b-a}{n}$ where $a = 0$, $b = 1$ and $n = 5$. Then, the candidates used the value of h and the given limits to construct a table of values for x^2e^x as shown below:

	y_0	y_1	y_2	y_3	y_4	y_5
x	0	0.2	0.4	0.6	0.8	1.0
x^2e^x	0	0.04886	0.23869	0.65596	1.42435	2.71828

Thereafter, they used the values from the constructed table together with the trapezoidal rule formula $A = \frac{h}{2}[y_0 + y_5 + 2(y_1 + y_2 + y_3 + y_4)]$ to estimate the value of $\int_0^1 x^2e^x dx$ to four significant figures and obtained 0.7454.

In part (c) (ii), the candidates demonstrated adequate knowledge of integration. They applied "ILATE" procedure so as to identify the function which will be u and dv. Then, the candidates let $u = x^2$ and differentiated it to obtain $du = 2xdx$ and $dv = e^x dx$ then integrated it and got $v = e^x$. Thereafter, recalled the formula of integration by parts which is $\int udv = uv - \int vdu$, and substituted u and dv as $\int_0^1 x^2e^x dx = [x^2e^x]_0^1 - \int_0^1 xe^x dx$ which led to $\int_0^1 x^2e^x dx = [x^2e^x]_0^1 - 2([xe^x]_0^1) + 2[e^x]_0^1$ and got 0.7183, the required answer. In part (c) (iii), they subtracted the actual value from approximated value of the integral $\int_0^1 x^2 e^x dx$ to compute the absolute error and obtained 0.0271. Extract 7.1 illustrates a correct response from one of the candidates.

7 (a) soln

condition for root between interval

$f(x_1) \cdot f(x_2) < 0$

Given $x^2 - 2x - 1 = 0$

$f(x_1) = x_1^2 - 2x - 1$

$f(2) = 4 - 4 - 1$

$f(x_1) = -1$

$f(x_2) = 3^2 - (3 \times 2) - 1$

$f(x_2) = 2$

$f(x_1) \times f(x_2) = 2 \times -1$

$f(x_1) \cdot f(x_2) = -2$

since $f(x_1) \cdot f(x_2) < 0$

$\therefore$ The root lies between $2 \leq x \leq 3$

Hence verified.

(b) soln

$f(x) = x^2 - 2x - 1$

secant formula

$$x_{n+2} = x_n - \frac{(x_{n+1} - x_n) f(x_n)}{f(x_{n+1}) - f(x_n)}$$

$f(x_1) = 2$, $f(x_0) = -1$

$$x_2 = x_0 - \frac{(x_1 - x_0) f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = 2 - \frac{(3 - 2) \times -1}{2 + 1} = 2 + 1/3$$

$x_2 = 2.333$

$$x_3 = x_1 - \frac{(x_2 - x_1) f(x_1)}{f(x_2) - f(x_1)}$$

$f(x_2) = -0.223$

7 (b)

$$x_3 = 3 - \frac{(2.333 - 3) \times 2}{(-0.223 - 2)}$$

$$x_3 = 2.400$$

$$x_4 = x_2 - \frac{(x_3 - x_2) f(x_2)}{f(x_3) - f(x_2)}$$

$$x_4 = 2.333 - \frac{(2.400 - 2.333)(-0.22}{f(x_3) + 0.223}$$

$$f(x_3) = -0.04$$

$$x_4 = 2.333 + \frac{(2.4 - 2.333)(0.22}{-0.04 + 0.223}$$

$$x_4 = 2.415$$

$$f(x_4) = (2.415)^2 - (2 \times 2.415) - 1$$

$$f(x_4) = 0.002$$

$$x_5 = x_3 - \frac{(x_4 - x_3) f(x_3)}{f(x_4) - f(x_3)}$$

$$x_5 = 2.400 - \frac{(2.415 - 2.400)(-0.04)}{0.002 + 0.04}$$

$$x_5 = 2.414$$

∴ The approximate root of the equation is $x = 2.414$ (3 dp)

7 (c) (i)

Let, $y = \int_0^1 x^2 e^x dx$

$$h = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$$

x	$y = x^2 e^x$	First + Last terms	middle terms
0.00	0.000	0.000	
0.20	4.886×10^{-2}		4.886×10^{-2}
0.40	2.387×10^{-1}		2.387×10^{-1}
0.60	6.560×10^{-1}		6.56×10^{-1}
0.80	1.424×10^{00}		1.424×10^{00}
1.00	2.718×10^{00}	2.718	
		2.718	2.368

$$A = \frac{1}{2} h \left[(1^{st} + last) + 2 \Sigma (\text{middle terms}) \right]$$

$$A = \frac{0.2}{2} \left[2.718 + (2 \times 2.368) \right]$$

$$A = 7.454 \times 10^{-1}$$

$$A = 0.7454$$

$$\therefore \int_0^1 x^2 e^x dx = 0.7454 = 7.454 \times 10^{-1}$$

(ii) Given,

$$\int_0^1 x^2 e^x dx$$

let $u = x^2$

$$\frac{du}{dx} = 2x$$

7 (c) (ii)

$$\int dv = \int e^x dx$$

$$v = e^x$$

$$\int_0^1 x^2 e^x dx = x^2 e^x - \int e^x (2x)\, dx$$

consider $\int e^x 2x\, dx$

let $u = x$

$$\frac{du}{dx} = 1$$

$$\int dv = \int e^x dx$$

$$v = e^x$$

$$2\int x e^x dx = 2\left[x e^x - \int e^x dx\right]$$

$$= 2\left[x e^x - e^x\right]$$

$$= 2e^x (x-1)$$

$$\int_0^1 x^2 e^x dx = x^2 e^x - 2e^x (x-1)$$

$$= \left[e - 2e(0)\right] - \left[0 + 2\right]$$

$$= e - 2$$

$$= 7.183 \times 10^{-1}$$

$$= 0.7183$$

$$\therefore \int_0^1 x^2 e^x dx = 0.7183 = 7.183 \times 10^{-1}$$

7 (c) (iii)

$$\text{error } (e) = \left| \text{Value 1} - \text{Value 2} \right|$$

$$e = \left| 0.7183 - 0.7454 \right|$$

$$e = 2.710 \times 10^{-2}$$

$\therefore$ The absolute error is 2.710×10^{-2}

Extract 7.1: A sample of correct responses to question 7

In Extract 7.1, the candidate showed great knowledge of numerical integration as he/she managed to use $f(x_1)f(x_2) < 0$ to judge the given root range in part (a). While in part (b), the candidate applied the Secant method properly to approximate the root for the given equation. And in part (c), the candidate used the trapezoidal method to evaluate the given definite integral and then calculated an absolute error correctly.

Minority of the candidates who attempted this question did not perform well due to inadequate knowledge in numerical integration. In part (b), some candidates used inappropriate formula to evaluate the root of the given function. For example, a few candidates used $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ which is Newton's Raphson formula instead of Secant method. In part (c) (i), some considered the number of strips as the number of ordinates. In part (c) (ii), some candidates failed to solve $\int_0^1 x^2 e^x dx$ using integration by part method. For example, they let $u = e^x$ and $v = x^2$ which led to an incorrect answer. Also, in part (c) (iii), some candidates had no knowledge of error. They used a wrong formula in computing absolute error, such as $\text{Absolute Error} = \text{Exact Value} - \text{Approximated Value}$. While others computed absolute error as $\text{Absolute error} = \frac{\text{Exact Value} - \text{Approx Value}}{\text{Exact Value}}$ instead of $\text{Absolute Error} = \left|\text{Exact. Value} - \text{Approx. Value}\right|$. Extract 7.2 shows parts (b) and (c) of a sample of an incorrect response from one of the candidates.

7 (b) solution

from $x^2 - 2x - 1 = 0$

from $f(x) = x^2 - 2x - 1$

$f(x_n) = x_n^2 - 2x_n - 1$

$f'(x_n) = 2x_n - 2$

from $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

for $n = 0$, $\cancel{x_1 = x_0}$

Know to get x_0.

$\frac{-1 + 2}{2} = 0.5$.

$n = 0$, $x_1 = x_0 - \frac{x_0^2 - 2x_0 - 1}{2x_0 - 2}$

$x_1 = 0.5 - \frac{(0.5)^2 - 1 - 1}{1 - 2}$

1st iteration, $x_1 = 1.75$.

~~when~~

when $n = 1$, $x_2 = 1.75 - \frac{(1.75)^2 - 2(1.75) - 1}{2(1.75) - 2}$

2nd iteration, $x_2 = \cancel{0.458}$ 2.708

when $n = 2$, $x_3 = 2.708 - \frac{(2.708)^2 - 2(2.708) - 1}{2(2.708) - 2}$

3rd iteration, $x_3 = 2.439$

7 (b) 4 $n=3$, $x_4 = 2.414 - \frac{(2.414)^2 - 2(2.414) - 1}{2(2.414) - 2}$

4^{th} iteration $x_4 = 2.414$.

7 (c) Given.

(i) $\int_0^1 x^2 e^x \, dx$.

soln:

$h = \frac{b-a}{n}$ $\quad$ n = number of strips − 1.

$n = 5 - 1$

$h = \frac{1-0}{4}$ $\quad$ $n = 4$.

$h = 0.25$

x	$y = x^2 e^x$	$y_n + y_0$	Middle term
0	0	0	
0.25	0.0802		0.0802
0.5	0.4122		0.4122
0.75	1.1908		1.1908
1	2.7183	2.7183	
Total		2.7183	1.6832

Area $= \frac{h}{2} \left[y_n + y_0 + 2\sum(\text{odd Middle}) \right]$.

$= \frac{0.25}{2} \left[2.7183 + 2(1.6832) \right]$.

$= 0.125 \left[6.0847 \right]$

Area $= 0.7606$.

7 c). (i) Trapezoidal rule $A = \frac{h}{2}\left[y_0 + y_n + 2 \sum \text{Middle term} \right]$

$= \frac{0.25}{2} \left[2.718281822 + 2(1.683244826 \right.$

$= 0.71828$

$\therefore$ Area of Trapezoidal rule $= 0.71828$

Extract 7.2: A sample of incorrect responses to question 7(b) and (c)

In Extract 7.2, the candidate used Newton's Raphson formula instead of Secant formula as per instruction in part (b) of the question. While in part (c), the candidate was not able to realize the difference between ordinates and strips leading him/ her to obtain the wrong value of h and hence failed to approximate the value of the given integral.

2.1.8 Question 8: Coordinate Geometry I

The question comprised three parts (a), (b) and (c). In part (a), the candidates were asked to determine the possible value of k when a perpendicular distance of point $(1, k)$ to the line $6x+9=8y$ is 5.5. In part (b), the candidates were required to find the tangent of the acute angle between the curves $y=2x^2-3$ and $y=x^2-5x+3$ which intersects at points U and W, of which W is in the fourth quadrant. Furthermore, in part (c), they were required to find the equation of a circle in the form of $x^2+y^2+2gx+2fy+c=0$ which passes through points $A(0,1)$, $B(4,3)$ and $C(1,-1)$.

The analysis indicates that the performance of the candidates who scored marks from 0 to 3 were 45.8 per cent, while 34.2 per cent scored 3.5 to 5.5 marks and 19.9 per cent scored 6 to 10 marks. Figure 9 presents the candidates' performance summary for this question.

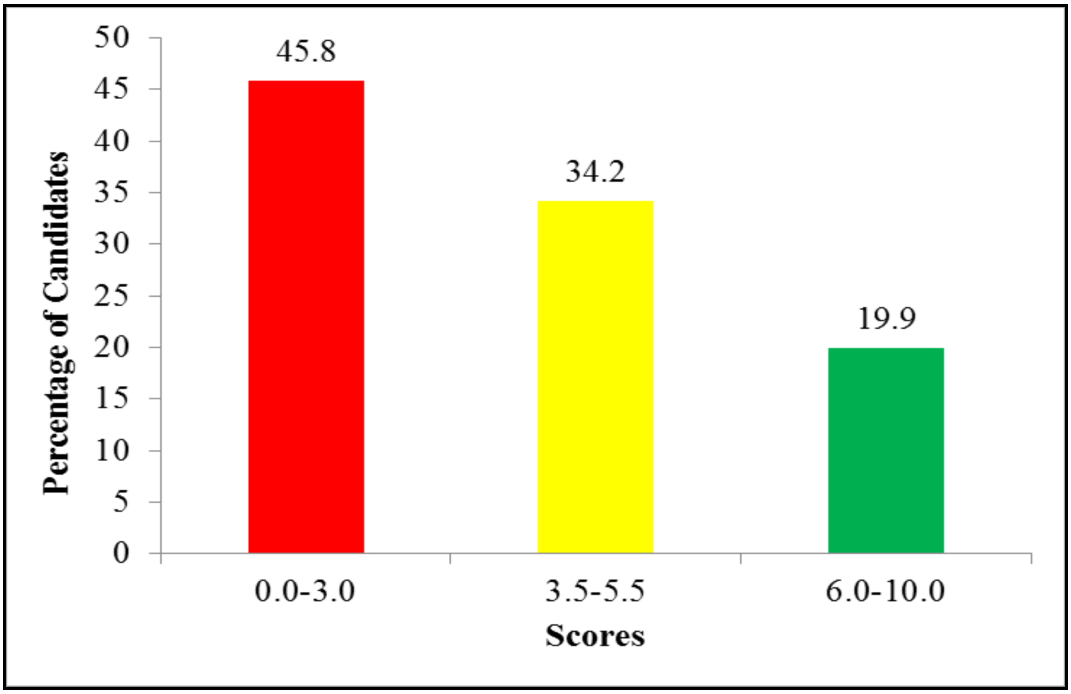


Figure 9: *Candidates' Performance in Question 8*

From Figure 9, it is observed that 54.1 per cent of the candidates scored 3.5 to 10 marks indicating that, the performance of the candidates in this question was average.

The candidates with good performance in the question demonstrated the following competences. Those who answered part (a) correctly were able to use the perpendicular distance formula $d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$ of a point from a line to determine the value of k. They were able to substitute the value of $x_1 = 1$ and $y_1 = k$ from point $(1, k)$ in the line $6x + 9 = 8y$ and $d = 5.5$ in a formula, then they obtained the values of $k = \frac{35}{4}$ or $k = -5$. In part (b), the candidates were also able to use the formula $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ to find the tangent between two curves. They determined the points of intersection and from which W is at $(1, -1)$ in the fourth quadrant. Thereafter, they substituted the value of slopes $m_1 = 4$ from curve $y = 2x^2 - 3$ and $m_2 = -3$ from $y = x^2 - 5x + 3$ to get $\tan\theta = \left|-\frac{7}{11}\right| = \frac{7}{11}$ which was the correct answer.

Finally, in part (c), most candidates were able to find the equation of a circle which passes through points $A(0,1)$, $B(4,3)$ and $C(1,-1)$ in the form of $x^2 + y^2 + 2gx + 2fy + c = 0$. They substituted these points A, B and C in the equation and formed the three equations, that is $1 + 2f + c = 0$, $25 + 8g + 6f + c = 0$ and $2 + 2g - 2f + c = 0$ respectively. By solving these equations they got $g = -\frac{5}{2}$, $f = -1$ and $c = 1$. Thus, they used these values to get the required equation of a circle which is $x^2 + y^2 - 5x - 2y + 1 = 0$. Extract 8.1 shows a good response from one of the candidates who performed well in this question.

8. (a) Perpendicular distance

$$d = \left| \frac{ax + by + c}{\sqrt{a^2 + b^2}} \right|$$

$$6x + 9 = 8y$$
$$6x - 8y + 9 = 0$$

$$d = \left| \frac{6x - 8y + 9}{\sqrt{6^2 + 8^2}} \right| \quad \text{at } (1, k)$$

$$5.5 = \left| \frac{6 - 8k + 9}{10} \right|$$

$$55 = |15 - 8k|$$

$15 - 8k = 55$ OR $-(15 - 8k) = 55$

$-8k = 40$ OR $8k = 55 + 15$

$k = -5$ OR $k = 35/4$

(b)

$$y = 2x^2 - 3$$
$$y = x^2 - 5x + 3$$
$$2x^2 - 3 = x^2 - 5x + 3$$
$$x^2 + 5x - 6 = 0$$
$$x = 1 \text{ OR } x = -6$$

when

$x = 1$ $\quad$ $x = -6$

$y = -1$ $\quad$ $y = 69$

$\therefore$ $U = (-6, 69)$ and $W = (1, -1)$

8 (b) for four quadrant

$X = 1$

$y = -1$

slope of curve $y = 2x^2 - 3$

$\frac{dy}{dx} = 4x$

$m_1 = 4$

slope of curve $y = x^2 - 5x + 5$

$\frac{dy}{dx} = 2x - 5$

$m_2 = -3$

$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_2 m_1}\right|$

$= \frac{-3 - 4}{1 + -12}$

$= \frac{-7}{-11}$

$= \frac{7}{11}$

$\therefore$ The tangent of angle is $\frac{7}{11}$.

8. (c) At A (0, 1)

$x^2 + y^2 + 2gx + 2fy + c = 0$

$0 + 1 + 0 + 2f + c = 0$

$2f + c = -1$ --- (i)

At B (4, 3)

$4^2 + 3^2 + 8g + 6f + c = 0$

$8g + 6f + c = -25$ --- (ii)

At C (1, −1)

$1^2 + (-1)^2 + 2g - 2f + c = 0$

$2g - 2f + c = -2$ --- (iii)

Solving eqn (i) (ii) and (iii)

$g = -5/2$

$f = -1$

$c = 1$

Required eqn of a circle is

$x^2 + y^2 + 2x(-5/2) + 2y(-1) + 1 = 0$

$x^2 + y^2 - 5x - 2y + 1 = 0$

Extract 8.1: A sample of correct responses to question 8

In Extract 8.1, part (a), the candidate was able to use the correct formula of distance to determine the value of k. While in part (b), the candidate found the tangent of the acute angle between curves correctly and finally in part (c), they wrote the equation of the circle which passes through three points given**.**

The analysis shows that, some candidates failed to reach to the required solution while attempting this question due to misconceptions and lack of knowledge. For instance in part (a), a few candidates failed to find the value of *k* as they were unable to substitute the values of a given point (*1, k*) and *d*=5.5 in a perpendicular distance formula of a point from a line that is $d=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$ to get $k=-5$ or $k=\frac{35}{4}$. For example, one among the candidates failed to understand the requirement of the question by finding the slope $m_1=\frac{6}{8}$ from the given line $6x+9=8y$. Likewise $m_2=\frac{-8}{6}$ using $m_1m_2=-1$ which is the condition for two lines to be perpendicular instead of determined the value of k using perpendicular distance formula. In part (b), they failed to find the point of intersection between two curves $y=2x^2-3$ and $y=x^2-5x+3$. So they got points $U(0,-3)$ and $W(-5,47)$ as their points of intersection instead of $U(-6,69)$ and $W(1,-1)$. Then, they used those wrong points to get the equation of the line $10x+y+3=0$, rather than finding the tangent of the acute angle at point W which was $\tan\theta=\left|-\frac{7}{11}\right|=\frac{7}{11}$. Others solved two curves simultaneously and obtained the equation of the line $10x+y-9=0$ as their answers for the tangent of the acute angle, which was an incorrect approach. Furthermore, some candidates were able to get those points of intersection which were $U(-6,69)$ and $W(1,-1)$, but they used those points to find the slope which was $\tan\theta=\frac{69--1}{-6-1}=-0.101576$ using the formula $\tan\theta=\frac{y_2-y_1}{x_2-x_1}$ which was a wrong approach and that led to a wrong answer.

In part (c), they substituted the given points $A(0,1)$, $B(4,3)$ and $C(1,-1)$ in the wrong expression of the circle which was $(x-x_1)(x-x_2)(x-x_3)+(y-y_1)(y-y_2)(y-y_3)$ instead of substituting the given points into the standard equation of the circle which is $x^2+y^2+2gx+2fy+c=0$. Then, solving for the values of g, f and c. Thereafter, they substituted those points in wrong expression and ended up with expression

$x^3 + y^3 - 5x^2 - 3y^2 + 4x - y + 3$ which was not the required equation. Extract 8.2 is a sample of an incorrect response from one of the candidates.

8 Given that

point $(9, k)$,

$6x + 9 = 8y$

Perpendicular distance $= 5.5$

From perpendicular distance.

$m_1 m_2 = -1$.

But $m_1 \Rightarrow 8y = 6x + 9$

$y = \frac{6x}{8} + \frac{9}{8}$.

$m_1 = \frac{6}{8}$.

At $m_2 =$

$m_1 m_2 = -1$

but $m_1 = \frac{6}{8}$.

$\frac{6}{8} m_2 = -1$

$m_2 = -\frac{8}{6}$.

8 b) Then

at $y = 2x^2 - 3$

for the values of x.

$y = -3$ or $y = 47$.

Then.

$m = \frac{\Delta y}{\Delta x}$

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{47 - -3}{-5 - 0}$

$m = -10$

Then

$m = \frac{\Delta y}{\Delta x}$ Using point $(0, -3)$

$-10 = \frac{y+3}{x-0}$

$-10x = y + 3$.

$y = -10x - 3$.

$-10x - y - 3 = 0$

$10x + y + 3 = 0$

$\therefore$ The tangent is $10x + y + 3 = 0$.

C

Solution

Given $A(0,1)$ $B(4,3)$ and $C(1,-1)$

Recall

$(x-x_1)(x-x_2)(x-x_3)+(y-y_1)(y-y_2)(y-y_3)$

then

$(x-0)(x-4)(x-1)+(y-1)(y-3)(y+1)$

$x^2-4x-0-0$

$(x^2-4x)(x-1)+(y^2-3y-y+3)(y+1)$

$x^3-x^2-4x^2+4x+y^3+y^2-3y^2-3y-y^2-y+3y+$

$x^3-5x^2+4x+y^3-2y^2-\cancel{3y}-y^2-y+\cancel{3y}+3$

$x^3-5x^2+4x+y^3-3y^2-y+3$

$x^3+y^3-5x^2-3y^2+4x-y+3$

Extract 8.2: A sample of incorrect responses to question 8

In Extract 8.2, the candidate did not realize the correct formula for the perpendicular distance of a point from a line to determine the value of k in part (a). While in part (b), the candidate applied the formula used to find the slope of a straight line instead of finding the tangent of an acute angle between the given curves. In part (c), the candidate failed to establish the equation of a circle passing through three points.

2.1.9 Question 9: Integration

Question 9 comprised parts (a) and (b). In part (a), the candidates were given $I_m=\int \sin^m x dx$ and required to show that $I_m=\frac{1}{m}\left(-\cos x\sin^{m-1}x+(m-1)I_{m-2}\right)$, and hence find the value of $\int \sin^3 xdx$. In part (b), the candidates were required to evaluate $\int_0^1 \frac{1}{(x+1)(x^2+2x+2)}dx$ and correct the answer in three decimal places.

The data analysis shows that among 13,448(99.99%) candidates who attempted the question 10,413(77.4%) candidates scored 0 to 3 marks, while 1,679(12.5%) candidates scored 3.5 to 5.5 marks and 1,356(10.1%) candidates scored 6 to 10 marks. The performance of candidates' summary is given in Figure 10.

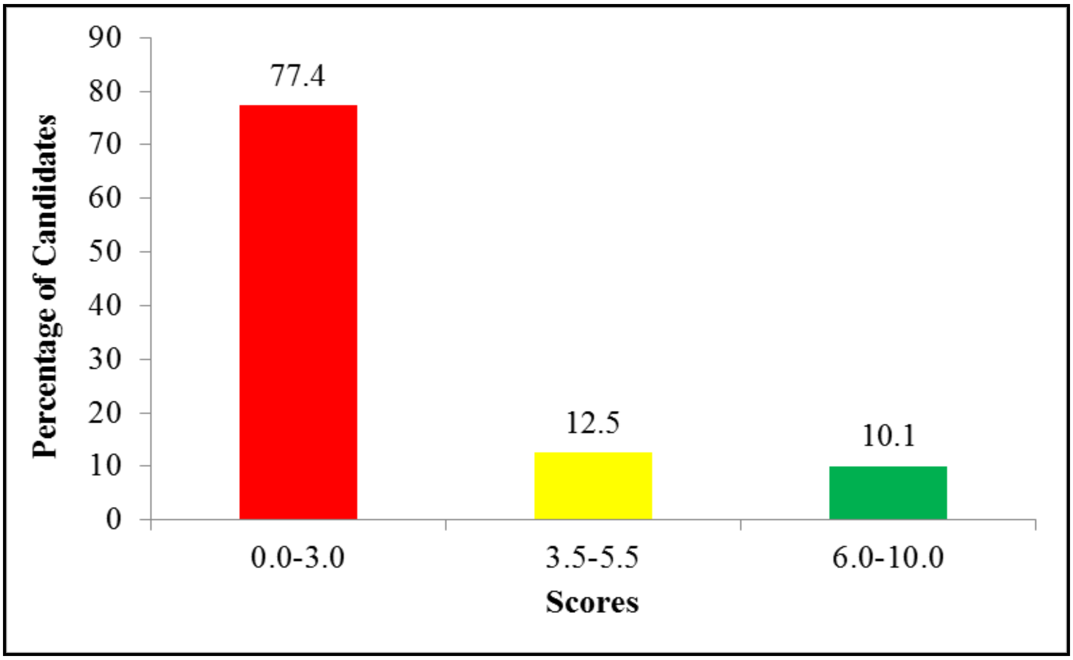


Figure 10: *Candidates' Performance in Question 9*

Generally, the candidates' performance in this question was weak as the percentage of candidates who scored 0 to 3 marks is high.

Most of candidates failed to answer this question correctly due to inadequate knowledge in integration techniques. For instance in part (a), some candidates failed to transform $\int \sin^m x dx$ into $\int \sin^{m-1} x \sin x dx$, instead they let $x = \pi - x$ from the given integral. Then, they used the substitution $u = \sin(\pi - x)$. Hence, they ended up with $m \cos x I_m = -\cos x \,\mathrm{Im} + \sin^m x \sin x$ although they concluded as $I_m = \frac{1}{m}\left(-\cos x \sin^{m-1} x + (m-1) I_{m-2}\right)$. Others failed to substitute the value of *m=3* by comparing $\sin^m x = \sin^3 x$ into $I_m = \frac{1}{m}\left(-\cos x \sin^{m-1} x + (m-1) I_{m-2}\right)$. Instead, they expressed $\sin^3 x$ into $(1 - \cos^2 x)\sin x$ and used the substitution $u = \cos x$. Therefore, they ended up with $\int \sin^3 x dx = -\cos x + \frac{\cos^3 x}{3} + c$.

In part (b), those who failed had insufficient knowledge on partilization and factorization. For example, instead of factorizing $x^2 + 2x + 2$ to get $(x+1)^2 + 1$, they got $(x+1)(x+1)$ and ended up with an incorrect answer as

$\int_0^1 \frac{dx}{(x+1)(x^2+2x+2)} = \int_0^1 \frac{dx}{(x+1)(x+1)(x+1)} = \frac{-3}{4}$. Also, a few candidates failed to choose a suitable substitution to make the given integral easier to evaluate. For instance, they used the substitution $t = x^2 + 2x + 2$ instead of using the substitution of $x + 1 = \tan\theta$ after factorizing $(x+1)(x^2+2x+2)$ into $(x+1)((x+1)^2+1)$ which led to an incorrect answer of 0.15. Extract 9.1 provides a sample of an incorrect response from one of the candidates in part (b) of the question.

9(b) Solution

Given

$\int_0^1 \frac{1}{(x+1)(x^2+2x+2)} dx = \int_0^1 \frac{dx}{(x+1)(x+1)(x+1)}$.

$= \int \frac{dx}{(x+1)^3}$

let

$u = x+1$

$du = dx$

$= \int_0^1 \frac{du}{u^3}$

$= \int_0^1 u^{-3} du$

$= \left[\frac{u^{-2}}{-2}\right]_0^1$

$= -\frac{u^{-2}}{2}$

But $u = x+1$

$= \left[\frac{(-x-1)^{-2}}{2}\right]_0^1$

$= \left[\frac{-1}{2(x+1)}\right]_0^1$

9(b) $\int \frac{dx}{(x+1)^3} = \left[-\frac{1}{2}(1+1) - \frac{1}{2}(0+1)\right]$

$= -\frac{1}{4} - \frac{1}{2}$

$= -\frac{3}{4}$

$\therefore \int_0^1 \frac{1}{(x+1)(x^2+2x+2)} dx = -\frac{3}{4}$

Extract 9.1: A sample of incorrect responses to question 9(b)

In Extract 9.1, the candidate failed to factorize the expression $x^2 + 2x + 2$, leading to integration of an incorrect integral and obtaining a wrong answer.

In spite of the majority of candidates who failed to attempt this question correctly, there were a few candidates who answered it correctly. Thus, the candidates who scored high marks on this question, in part (a) were able to show step by step the reduction formula for sine by starting transforming $\int \sin^m x dx$ into $\int \sin^{m-1} x \sin x dx$. Then by using the integration by parts technique by letting $u = \sin^{m-1} x$ and $dv = \sin x dx$. Thereafter, they ended up with equation $I_m = \frac{1}{m}\left(-\sin^{m-1} x \cos x + (m-1) I_{m-2}\right)$ which was required to show where m is any integer. Thereafter, they used it to find $\int \sin^3 x dx$, by substituting the value of $m = 3$ and obtained $I_3 = \frac{1}{3}\left(-\sin^2 x \cos x - 2\cos x\right) + c$.

For part (b), they were able to integrate the integral based on partilization technique by letting $\frac{1}{(x+1)(x^2+2x+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+2}$ in order to decompose it and obtained the value of $A = 1$, $B = -1$ and $C = -1$. Finally they integrated $\int_0^1 \frac{1}{x+1} dx - \int_0^1 \frac{x+1}{x^2+2x+2} dx$ and got the correct answer of 0.235. Extract 9.1 shows one of a correct response from one of the candidates.

9(a)	Soln	
	$I_m = \int \sin^m x \, dx$	
	$= \int \sin x \, \sin^{m-1} dx$	
	$u = \cancel{\sin m-1} \sin^{m-1} x$	
	$du = (m-1)(\sin^{m-2} x)(\cos x) \, dx$	

$V = \int \sin x = -\cos x$

$\int u\,dv = uv - \int u\,dv$

$I_m = -\cos x \sin^{m-1} x - \int -\cos x (\cos x)(m-1)\sin^{m-2} x\,dx$

$I_m = -\cos x \sin^{m-1} x + m-1 \int \sin^{m-2} x (\cos^2 x)\,dx$

$= -\cos x \sin^{m-1} x + (m-1) \int \sin^{m-2} x (1 - \sin^2 x)\,dx$

$I_m = -\cos x \sin^{m-1} x + (m-1)\left(\int \sin^{m-2} x - \int \sin^m x\,dx\right)$

$I_m = -\cos x \sin^{m-1} x + (m-1)(I_{m-2} - I_m)$

$I_m + I_m(m-1) = -\cos x \sin^{m-1} x + (m-1) I_{m-2}$

$m I_m = -\cos x \sin^{m-1} x + (m-1) I_{m-2}$

$I_m = \frac{1}{m}\left(-\cos x \sin^{m-1} x + (m-1) I_{m-2}\right)$

for $m = 3$

$= \frac{1}{3}\left(-\cos x \sin^2 x + 2 \int \sin x\,dx\right)$

$\int \sin^3 x = \frac{1}{3}\left(-\cos x \sin^2 x + {}^{-}2\cos x\right) + C$

9(b) Sol

$\int_0^1 \frac{1}{(x+1)(x^2+2x+2)}\,dx$

te $\frac{1}{(x+1)(x^2+2x+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+2}$

$1 = A(x^2+2x+2) + (Bx+C)(x+1)$

put $x = -1$

$1 = A(1-2+2)$

$A = 1$

put $x = 0$

$1 = 2A + C$

$C = -1$

Comparing coeffs

$0 = A + B$

$B = -1$

$$\therefore \int \frac{1}{(x+1)(x^2+2x+2)}\,dx = \int \frac{1}{x+1}\,dx - \int \frac{x+1}{x^2+2x+2}\,dx$$

$$= \ln(x+1) - \int \frac{x+1}{\cancel{2x} - x^2+2x+2}\,dx$$

let $u = x^2+2x+2$

$$\frac{du}{2} = \frac{\cancel{2}(x+1)\,dx}{\cancel{2}}$$

$$\int \frac{1}{u}\frac{du}{2} = \frac{1}{2}\ln(u)$$

$$= \left[\ln(x+1) - \frac{1}{2}\ln(x^2+2x+2)\right]_0^1$$

$$= \left[\ln(1) - \frac{1}{2}\ln(5)\right] - \left(\ln(1) - \frac{1}{2}\ln(2)\right)$$

$$\therefore \int_1^0 \frac{1}{(x+1)(x^2+2x+2)}\,dx = 0.235$$

Extract 9.2 A sample of correct responses to question 9

In Extract 9.2, in part (a), the candidate was able to use the techniques of integration by parts to show the required reduction formula of $\int \sin^m x dx$. Then, he/she used the obtained formula to integrate $\int \sin^3 x dx$. In part (b), the candidate was able to evaluate the value of definite integral correctly to three decimal places.

2.1.10 Question 10: Differentiation

The question had parts (a), (b) and (c). In part (a), the candidates were required to differentiate $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ with respect to *x*. In part (b) (i), the candidates were required to use Maclaurin's theorem to find a series expansion of $\ln(1+x)$ up to the term containing x^4. In (b) (ii), they were required to use the results from expansion of $\ln(1+x)$ by using Maclaurin's theorem to compute $\ln(1.02)$ to four decimal places.

Furthermore, in part (c), the candidates were given the function $f(x, y) = \sin xy$, and were required to find $\frac{\partial^2 f}{\partial x \partial y}$.

The analysis of data shows that, among 13,441 (99.9%) candidates who answered the question, 8,570 (63.7%) candidates scored 0 to 3 marks as shown in Figure 11. A total of 2,204 (16.4%) candidates scored 3.5 to 5.5 marks and 2,667 (19.8%) candidates scored 6 to 10 marks.

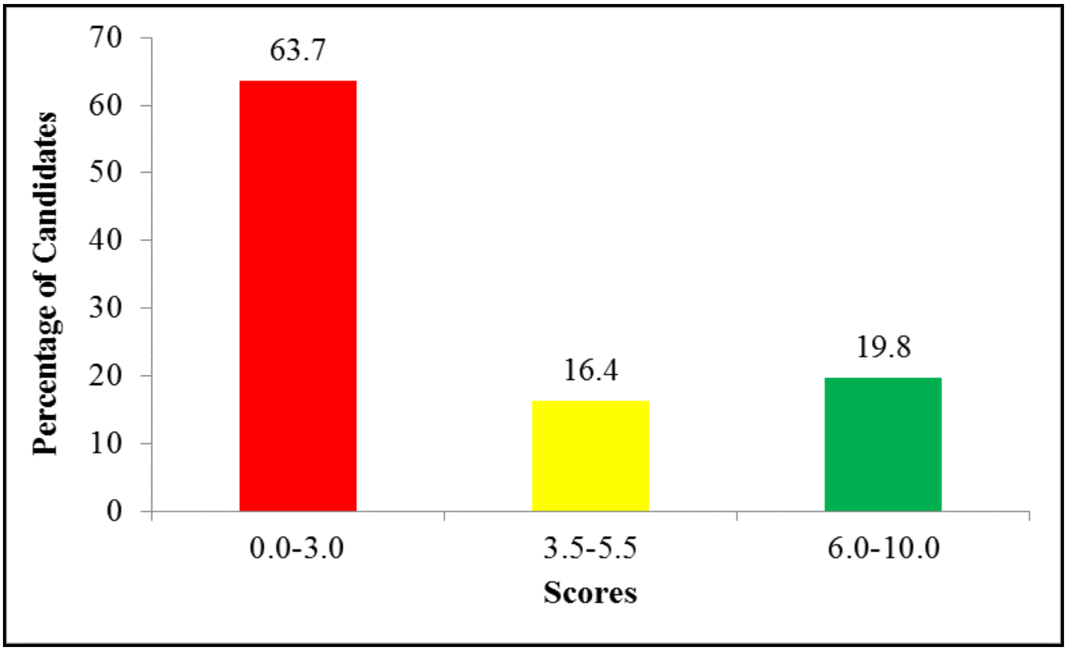


Figure 11: *Candidates' Performance in Question 10*

The candidates' performance summary in Figure 11 identifies that, the performance is average as 36.2 per cent of the candidates scored from 3.5 to 10 marks.

Candidates with good performances in part (a) show good ability to differentiate the inverse of cosine and use the quotient rule technique $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ to differentiate $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ with respect of x. They considered $u = 1 - x^2$ and $v = 1 + x^2$ from $\cos y = \frac{1-x^2}{1+x^2}$. Then, they got $\frac{dy}{dx} = \frac{2}{1+x^2}$. In part (b) (i), the candidates were able to use Maclaurin's theory $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f''''(0)$ to find the series expansion of $\ln(1+x)$ up to the term containing x^4 and obtained $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$. In part (b) (ii), they were able to use the results obtained in part (b) (i) to compute $\ln(1.02)$ to four decimal places by transforming $\ln(1.02)$ into $\ln(1+0.02)$ where $x = 0.02$. Then, they substituted $x = 0.02$ into $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$ as $\ln(1.02) = 0 + 0.02 - \frac{(0.02)^2}{2} + \frac{(0.02)^3}{3} - \frac{(0.02)^4}{4}$ to get $\ln(1.02) = 0.0198$ to four decimal places.

Lastly, in part (c), they were able to differentiate first with respect to y as $\left(\frac{\partial f}{\partial y}\right) = x\cos xy$ and second with respect to x as $\frac{\partial^2 f}{\partial x \partial y} = \cos xy - xy\sin xy$. Extract 10.1 shows a sample of a correct response from one of the candidates.

10. (a) $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Differentiating

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \cdot \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$= \frac{-(1+x^2)}{\sqrt{1+2x^2+x^4-1+2x^2-x^4}} \cdot \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$= \frac{-(1+x^2)}{\sqrt{4x^2}} \cdot \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$= \frac{-(1+x^2)}{2x} \cdot \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$= \frac{-(1+x^2)}{2x} \cdot \left[\frac{(1+x^2)(-2x) - (1-x^2)(2x)}{(1+x^2)^2}\right]$$

$$= \frac{-(1+x^2)}{2x} \cdot \left[\frac{-2x-2x^3-2x+2x^3}{(1+x^2)^2}\right]$$

$$= \frac{-(1+x^2)}{2x}\left[\frac{-4x}{(1+x^2)^2}\right]$$

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

(b) (i) $f(x) = \ln(1+x)$ $\quad f(0) = 0$

$f'(x) = \frac{1}{1+x}$ $\quad f'(0) = 1$

$f''(x) = -(1+x)^{-2}$ $\quad f''(0) = -1$

$f'''(x) = 2(1+x)^{-3}$ $\quad f'''(0) = 2$

$f^4(x) = -6(1+x)^{-4}$ $\quad f^4(0) = -6$

10. (b) From Maclaurin's theorem

$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \frac{x^4 f^{4}(0)}{4!} + \cdots$

$\ln(1+x) = 0 + x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} + \cdots$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

(ii) $\ln(1.02) = \ln(1+0.02)$

$x = 0.02$

$\ln(1+0.02) = 0.02 - \frac{0.02^2}{2} + \frac{0.02^3}{3} - \frac{0.02^4}{4} + \cdots$

$\ln(1.02) = 0.0198$

(c) $f(x,y) = \sin xy$

$\frac{\partial f}{\partial y} = x\cos xy$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$= \frac{\partial}{\partial x}(x\cos xy)$

$= \cos xy + x(-y\sin xy)$

$\frac{\partial^2 f}{\partial x \partial y} = \cos xy - xy\sin xy$

Extract 10.1: A sample of correct responses to question 10

In Extract 10.1, the candidate had enough skills to differentiate the inverse of cosine in part (a). Whereas in part (b), the candidate established Maclaurin's series of logarithmic functions to find the series correctly. Then he/she used it to compute $\ln(1.02)$ properly. In part (c), the candidate applied the concept of partial derivative to differentiate the given function.

Furthermore, the analysis shows that there were candidates who performed this question poorly as they faced some challenges. In part (a), some candidates used inappropriate rule to differentiate the inverse of cosine. For instance, some of them failed to use the quotient rule after taking $u = \frac{1-x^2}{1+x^2}$ from $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$. Instead, they differentiated using the product rule as $\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)=\frac{udv}{dx}+\frac{vdu}{dx}$ and they ended up with an incorrect derivative which was $\frac{dy}{dx}=\frac{-\left(-2x^3+x^2+1\right)}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}}$.

For part (b), they failed to get the correct values of $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$ and $f''''(0)$ in Maclaurin's expansion of $\ln(1+x)$ hence got the wrong values of 1, $\frac{1}{2}$, $\frac{-1}{4}$, $\frac{1}{4}$ and $\frac{-1}{4}$ respectively, so they ended up with the equation $\ln(1+x)=1+\frac{1}{2(1+x)}-\frac{2}{4\left(1+x^4\right)}+\frac{3}{1+x^3}$ rather than getting the values of $f(0)=0$, $f'(0)=1$, $f''(0)=-1$, $f'''(0)=2!$ and $f''''(0)=-3!$ to get the correct or the required Maclaurin's expansion of $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}$. Others failed to transform $\ln(1.02)$ into $\ln(1+0.02)$ to get $x=0.02$ and substituted it into the incorrect Maclaurin's expansion and got $\ln(1.02)=0.07189$ which was a wrong answer.

In part (c), most candidates failed to understand how to find the second partial derivatives. For instance, the candidates expressed $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial f}{\partial x}\times\frac{\partial f}{\partial y}$ instead of writing $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$. So, the candidates ended up with an incorrect answer which was $\frac{\partial^2 f}{\partial x\partial y}=xy\sin^2 xy$. Extract 10.2 shows a sample of one of the incorrect responses from one of the candidates.

10. a) $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ soln

$\dfrac{dy}{dx} = ?$

from $\cos^{-1}(x) = -\dfrac{1}{\sqrt{1-x^2}}$

so, $\dfrac{dy}{dx}$ of $\dfrac{1-x^2}{1+x^2} = \cancel{uv^2}\ u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

$= 1-x^2 \cdot 2x + 1+x^2 + (-2x)$

$\dfrac{d\left(\frac{1-x^2}{1+x^2}\right)}{dx} = 2x - 2x^3 + 1 + x^2 - 2x$

$\dfrac{dy}{dx} = \left(\cancel{2x} - 2x^3 + 1 + x^2 - \cancel{2x}\right) \cdot -\dfrac{1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}}$

$\therefore \dfrac{dy}{dx} = -\dfrac{\left(-2x^3 + x^2 + 1\right)}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}}$

b) i) Soln

from $f(x) = f(0) + hf'(0) + \frac{h^2 f''(0)}{2!} + \ldots$

$f(x) = \ln(1+x)$

$f'(x) = \frac{1}{1+x}$

$f''(x) = 2+x$

$f'''(x) = 1$

$f''''(x) = 0$

$f(x) = \ln 1 + h + \frac{2h^2}{2!} + \frac{h^3}{3!}$

$\therefore f(x) = \ln 1 + h + 2h^2/2! + h^3/3!$

c) Soln.

$f(x,y) = \sin xy$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y}$

$\frac{\partial f}{\partial x} = y \sin xy$

$\frac{\partial f}{\partial y} = x \sin xy$

so, $\frac{\partial^2 f}{\partial x \partial y} = y \sin xy \times x \sin xy$

$\therefore \frac{\partial^2 f}{\partial x \partial y}$ ~~$= xy \sin^2 xy$~~ $= xy \sin^2 xy$

Extract 10.2: A sample of incorrect responses to question 10

In Extract 10.2, in part (a), the candidate lacked skills on how to differentiate the inverse of cosine by applying an incorrect formula of quotient rule. While in part (b), the candidate failed to establish Maclaurin's series of logarithmic functions. In part (c), the candidate failed to apply the concept of partial derivative to differentiate the given function.

2.2 142/2 ADVANCED MATHEMATICS 2

2.2.1 Question 1: Probability

This question had three parts, (a), (b) and (c). Part (a) read as follows: The time taken by John to deliver milk to the High Street is normally distributed with mean 12 minutes and standard deviation 2 minutes. Candidates were required to estimate the number of days during the year when he takes longer than 17 minutes if he delivers milk every day (1 year = 365 days). In part (b), the question was written: Suppose that a group of people in a village attending hospital has been categorized according to the incidence of two diseases.

Sex	Malaria	Typhoid
Male	16	12
Female	12	10

Candidates were required to find the probability that a person chosen is a female given that the person is suffering from Malaria. In part (c), they were required to find how many ways can a hand of 4 cards be chosen from an ordinary pack of 52 playing cards?

The analysis of data shows that a total of 13,443(99.96%) candidates attempted the question on which 6,047(45.0%) candidates scored 0 to 5 marks. While 3,266(24.3%) candidates scored 5.5 to 8.5 marks and 4,130(30.7%) candidates scored 9 to 15 marks. Figure 12 presents a summary of candidates' performance in this question.

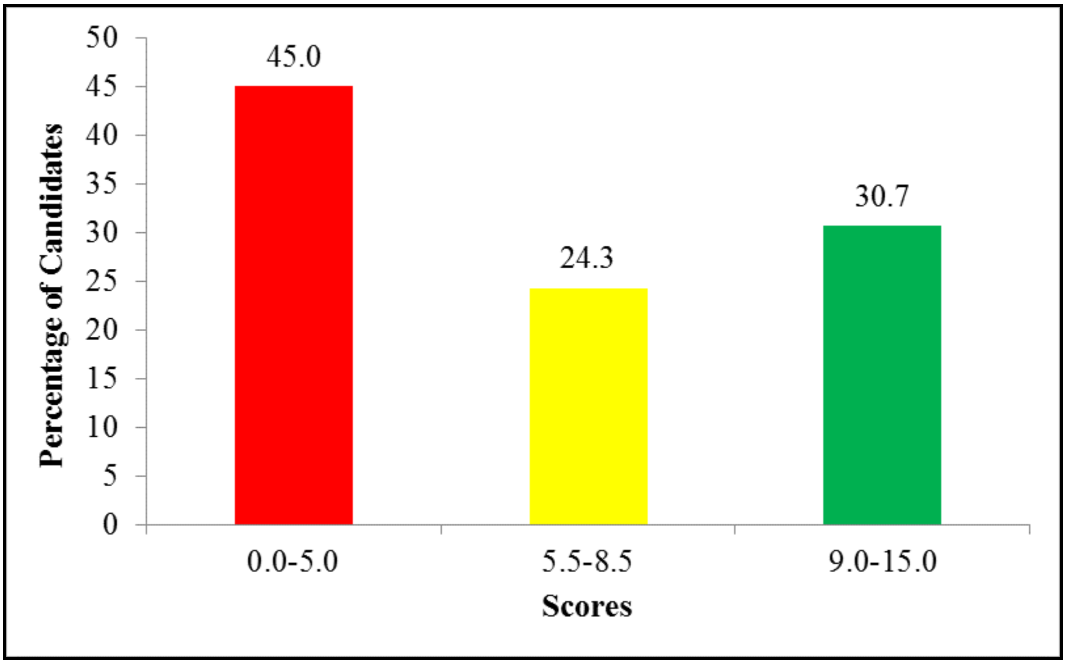


Figure 12: *Candidates' Performance in Question 1*

Furthermore, the analysis details that the performance of candidates in this question was average since 45 per cent of the candidates scored less than 5.5 marks.

The candidates who attempted this question correctly showed good understanding and skills on probability. In part (a), some candidates who performed well were able to standardize the normal random variable x into a Z score and obtained $Z = 2.5$ using the formula $Z = \frac{X-\mu}{\delta}$, where $\mu = 12$ and $\delta = 2, X = 17$. Moreover, they were able to use the scientific calculator to obtain $p\left(z > 2.5\right) = 0.00621$. Finally, they calculated the required number of days by multiplying the probability obtained by 365 days in the year, that is number of days = $365(0.00621) = 2$ days.

In part (b), the candidates used the correct formula for conditional probability, $P(F/M) = \frac{P(F \cap M)}{P(M)}$ with a correct substitution of $P(F \cap M) = \frac{12}{50}$ and $P(M) = \frac{28}{50}$ which gave the answer $P(F/M) = \frac{3}{7}$. On the other hand, other candidates were able to use an alternative method by identifying the number of sample space $n(S)$ and number of events $n(E)$ with values 28 and 12 respectively. They further used the formula $P(F/M) = \frac{n(E)}{n(S)}$ to obtain $P(F/M) = \frac{3}{7}$. In part (c), the candidates were able to recall and use the correct formula for combination, ${}^{n}C_{r} = \frac{n!}{(n-r)!r!}$ where $n = 52$ and $r = 4$. Then, they obtained ${}^{52}C_{4} = 270725$ as the number of ways of choosing a hand of 4 cards from an ordinary pack of 52 playing cards. Extract 11.1 shows one of correct responses from the candidates who attempted this question.

1 a) Soln

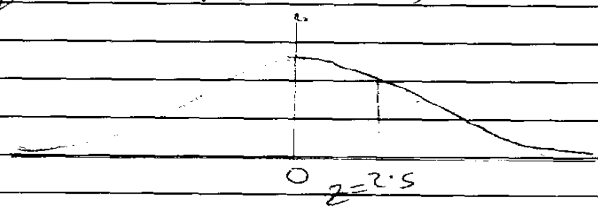
mean $(\mu) = 12$

S.D $(\sigma) = 2$

$Z = \frac{X - \mu}{\sigma}$

$Z = \frac{17 - 12}{2} = 2.5$

required $P(Z > 2.5)$



Prequired $= 6.21 \times 10^{-3}$

Number of days $= 6.21 \times 10^{-3} \times 365$

$= 2.27 \approx 2$ days

$\therefore$ The number of days $= 2$ days

b) Soln

required P(

let M - malaria

m' - male

F - female

T - Typhoid.

required $P(F/m) = \frac{P(F \cap m)}{P(m)}$

1(b) $P(F \cap m) = \frac{12}{50}$

$P(m) = \frac{28}{28+22}$

$P(m) = \frac{28}{50} = \frac{14}{25}$

$P(F/m) = \frac{\frac{12}{50}}{\frac{14}{25}}$

$P(F/m) = \frac{3}{7}$

The probability is $= 3/7$

c) No of ways $= {}^{52}C_4$

Number of ways $= 270725$ ways

Extract 11.1: A sample of correct responses to question 1

In Extract 11.1, the candidate was well equipped with knowledge and skills of normal random variable, conditional probability and the fundamental principle of counting, which enabled him/her to attempt this question correctly.

Despite the good performance on this question there were some candidates who faced challenges while attempting the question. In part (a), some candidates were able to write $Z = \frac{X-\mu}{\delta}$ correctly but wrongly substituted $\mu = 17$, $\delta = 2$ and $X = 12$ to obtain $p(z > -2.5)$ instead of $p(z > 2.5)$. Again, it was observed that other candidates substituted $\mu = 12$, $\delta = 2$ and $X = 365$ which led to $p(z > 176.5)$. It was further observed that a few candidates used the poison distribution formula $p(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$. Then, they substituted $\lambda = 12$ and $x = 17$ to obtain $p(x = 17) = 0.3832471$ instead of $p(x > 17) = 0.00621$. Moreover, it was observed that a few candidates used the variance formula, $Var(x) = Ex^2 - (Ex)^2$ and substituted $\sigma^2 = Var(x) = 2$ to solve the equation by writing $4 = 12x^2 - (12x)^2$ which was not the equation as per the correct procedure. It was again noted that some candidates calculated the number of days n using these formulas $\bar{x} = pq$ and $\sigma = npq$, with substitution of $\bar{x} = 12$ $\sigma = 2$ respectively. Then, they solved simultaneously and obtained *n=6*. Finally, they calculated the number of days by multiplying 17 days by $\frac{1}{6}$ to obtain 2.833 days.

In part (b), some candidates tried to use the technique of the tree diagram but ended up with an incorrect probability of $\frac{3}{11}$ because the tree diagram needed to use the concept of Bayer's theorem. Again, it was noted that a few candidates recalled $P(F/M) = \frac{P(F \cap M)}{P(M)}$ correctly but made mistakes in obtaining the value of $P(F \cap M)$. For example, some wrote $\frac{12}{28}$ instead of $\frac{12}{50}$. Others obtained $P(F \cap M)$ using the formula $P(F \cap M) = P(F) + P(M) - P(F \cup M)$ with $P(F) = \frac{12}{22}$, $P(M) = \frac{28}{50}$ and $P(F \cup M) = \frac{12}{12}$ to get $P(F \cap M) = 0.10545$ instead of $\frac{6}{25}$.

In part (c), a few candidates used inappropriate formulae, for example they calculated the number of ways of choosing 4 cards in the pack of 52 cards using the concept of permutation as ${}^{n}P_{r} = \frac{n!}{(n-r)!}$ where $n = 52$ $r = 4$ to obtain 6497400. Also, there were candidates who wrote n^r where $n = 52$ and $r = 3$ to obtain 140608. Other candidates divided 52 by 4. This result was multiplied by four factorials to get 312 ways of choosing a hand of four from a pack of 52 playing cards. Extract 11.2 shows a sample of incorrect responses from one of the candidates who attempted the question.

1 a

Mean = 12

Standard deviation = 2

from

~~Mean (λ) =~~ $\cancel{P_r = \frac{\lambda^x e^{-\lambda}}{x!}}$

~~but x = 17~~

$$\text{Mean }(\lambda) = np$$

$$\lambda = np$$

$$12 = np$$

But

$$\sigma = \sqrt{npq}$$

$$2 = \sqrt{npq}$$

$$4 = npq$$

but $np = 12$

$$4 = 12q$$

$$q = \frac{1}{3}$$

But $p + q = 1 \Rightarrow p = 1 - q$

$$p = 1 - \frac{1}{3} \Rightarrow \frac{2}{3}$$

1 a

$\therefore p = \frac{2}{3}$

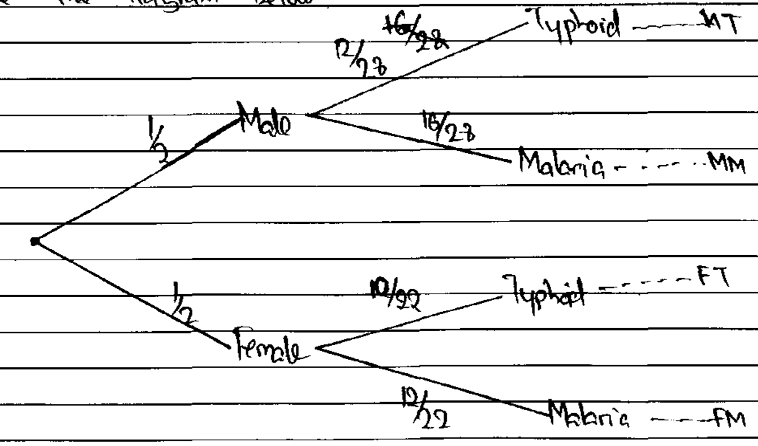
$q = \frac{1}{3}$

Since q = no of day he takes longer than 17 minutes.

$$q = \frac{1}{3} \times 365 \text{ days}$$

$$q = 121\tfrac{2}{3} \text{ days}$$

$\therefore$ No of days $= 121\frac{2}{3}$ days b Consider the tree diagram below.



Then;

P (female with malaria) $= FM$

$= \frac{1}{2} \times \frac{12}{22}$

$\Rightarrow$ The probability of female with malaria $= \frac{3}{11}$

1c Pack of cards $= 52$ (n)

no of card on hand $= 4$ (r)

$\therefore$ For number of ways.

For Case ①

(without repetition)

$^{n}P_{r} = \frac{n!}{(n-r)!}$

$^{n}P_{r} = \frac{52!}{(52-3)!}$

$^{n}P_{r} \Rightarrow 132,600$ ways.

$^{n}P_{r} \Rightarrow 132,600$ ways

Extract 11.2*:* A sample of incorrect responses to question 1

In Extract 11.2, part (a), the candidate failed to use the Z score formula to estimate the number of days. While in part (b), the candidate failed to use the formula for conditional probability. Also in part (c), he/she applied permutation instead of using the techniques of combination to solve the problem.

2.2.2 Question 2: Logic

This question consisted of three parts, (a), (b) and (c). In part (a), the candidates were required to write the converse and inverse of the statement "If you score an A grade in Logic test, then I will buy you a new car" in words and symbolic form. Part (b), required the candidates to use the truth table to examine $[(\sim p)\to(\sim q)]\wedge(p\to q)$ is equivalent to $(q\to p)\wedge(p\to q)$. In part (c), they were required to use laws of algebra of propositions to simplify $[p\wedge(p\vee q)]\vee[q\wedge(\sim(p\wedge q))]$.

The data analysis depicts that out of 13,443(99.96%) candidates who answered the question, 12,191(90.7%) candidates scored 9 to 15 marks. While 1,036(7.7%) candidates scored 5.5 to 8.5 marks and 216(1.6%) candidates scored 0 to 5 marks. The summary of the candidates' performance is given in Figure 13.

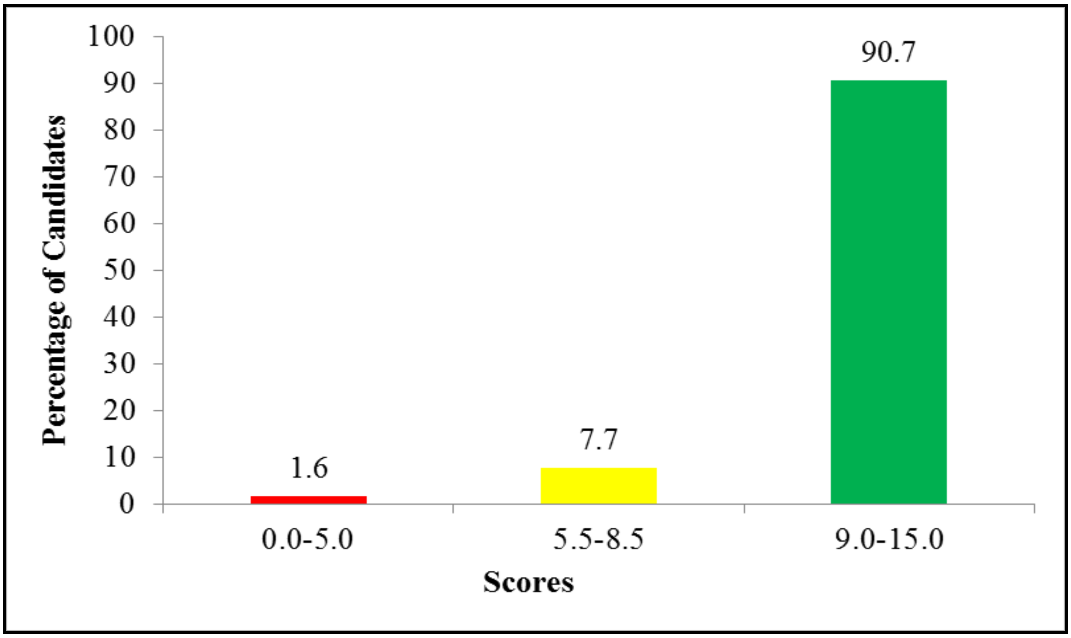


Figure 13: *Candidates' Performance in Question 2*

The performance of the candidates was good as observed in Figure 13 where 98.4 per cent of the candidates who attempted this question scored 5.5 to 15 marks.

The candidates who attempted this question correctly in part (a) were able to break down the conditional statement into premises and conclusion by letting p "You score an A grade in logic test" and q "I will buy you a new car" respectively. Finally, the candidates wrote the converse into words as "If I buy you a new car then you will score an A grade in logic test.", and in symbolic form is $q\to p$. Likewise, the inverse was written as “If you do not score an A grade in logic test, then I will not buy you a new car”. Thereafter, the candidates wrote it into symbolic form as $\sim q \rightarrow \sim p$.

In part (b), the candidates were able to arrive at the correct answer by following all the necessary steps for filling in the truth table and concluded that the two propositions $[(\sim p)\rightarrow(\sim q)]\wedge(p\rightarrow q)$ and $(q\rightarrow p)\wedge(p\rightarrow q)$ are logically equivalent. In part (c), the candidates who attempted it correctly had the knowledge of using laws of algebra of propositions such as distributive, complement, De-Morgan and absorption to simplify the compound statement $[p\wedge(p\vee q)]\vee[q\wedge(\sim(p\wedge q))]$ to $p\vee q$. Extract 12.1 illustrates a correct response from one of the candidates who attempted the question.

2. a) Let, p be you score an A grade in a logic test
q be buy you a new car

Statement form: p→q

converse (q→p) : If I buy you a new car then you will score an A grade in your logic test

inverse (~p→~q) : If you do not score an A grade in your logic test then I will not buy you a new car.

b)

p	q	~p	~q	(a) ~p→~q	(b) p→q	(c) q→p	a∧b	c∧b
T	T	F	F	T	T	T	T	T
T	F	F	T	T	F	T	F	F
F	T	T	F	F	T	F	F	F
F	F	T	T	T	T	T	T	T

The last two columns are equal thus [(~p→~q)] ∧ (p→q) is equivalent to (q→p) ∧ (p→q)

2.	(i) Solu:	
	$[p\wedge(p\vee q)]\vee[q\wedge(\sim(p\wedge q))]$ — given	
	$[(p\vee F)\wedge(p\vee q)]\vee[q\wedge(\sim(p\wedge q))]$ — distributive law.	
	$[p\vee(F\wedge q)]\vee[q\wedge(\sim(p\wedge q))]$ — Associative law.	
	$(p\vee F)\vee[q\wedge(\sim(p\wedge q))]$ — Identity law.	
	$p\vee[q\wedge(\sim(p\wedge q))]$ — Identity law	
	$p\vee[q\wedge(\sim p\vee\sim q)]$ — Demorgan's law.	
	$p\vee[(q\wedge\sim p)\vee(q\wedge\sim q)]$ — Distributive law.	
	$p\vee[(q\wedge\sim p)\vee F]$ — Complement law.	
	$p\vee(q\wedge\sim p)$ — Identity law.	
	$(p\vee q)\wedge(p\vee\sim p)$ — Distributive law.	
	$(p\vee q)\wedge T$ — Complement law.	
	$p\vee q$ — Identity law.	

Extract 12.1: A sample of correct responses to question 2

In Extract 12.1, the candidate was able to write the converse and inverse of a given statement both symbolically and in words. Furthermore, he/she was able to draw the correct truth tables and lastly he/she used laws of algebra of propositions in simplifying the compound statement.

Majority of the candidates scored high marks in this question. This shows that the candidates had adequate knowledge and skills on the concepts of logic. However, there were a few candidates who performed poorly on this question. For instance, in part (a), some were able to write the converse and inverse $q \to p$ and $\sim q \to \sim p$ respectively in symbolic form but incorrectly translated into words. For example, the converse into words was written "I will buy you a new car then if you score an A grade in a logic test". Others, wrote "I will buy a new car if you score an A grade in a logic test". On the other side, for the inverse in words, a few of candidates wrote "If you do not score an A grade in a logic test, then I will not buy you a new car". It was again noted that a number of candidates did not start with the word "If" in the beginning of a converse statement. Instead they wrote, "Unless you score an A grade in logic test, then I will not buy you a new car".

In part (b), the majority had no skills for preparing the truth table by using logical connectives. Some of them wrote "implication of F and F is F, F and T is F" which was incorrect. Lastly, in part (c), there were candidates who simplified $[p\wedge(p\vee q)]\vee[q\wedge(\sim(p\wedge q))]$ by using a set and logical symbols interchangeably. For example, they wrote $p \vee \sim p = \mu$. Also, there were candidates who introduced the identity $q \vee F$ instead of doing the distribution of $q \wedge (\sim p \vee \sim q)$ that led to incorrect steps. It was further seen in the responses that some faced difficulties in simplifying $p \wedge (p \vee q)$ and wrote $(p \vee q) \wedge (p \vee q)$. Extract 12.2 shows an incorrect response from one of the candidates.

2. a. let p = Score an A grade in a logic test.
q = Buy you a new car.

then;

Sentence : $p \longrightarrow q$

Converse

$q \longrightarrow p$.

- I will buy you a new car if you score an A grade in logic test.

Inverse.

$\sim p \longrightarrow \sim q$

- Unless you score an A grade in a logic test, then I will not buy you a new car.

2(b) Given: $[(\sim p) \rightarrow (\sim q)] \wedge (p \rightarrow q)$ and $(q \rightarrow p) \wedge (p \rightarrow q)$

Soln.

for $[(\sim p) \rightarrow (\sim q)] \wedge (p \rightarrow q)$

Truth table

p	q	~p	~q	~p → ~q (a)	p → q (b)	a ∧ b
T	T	F	F	F	F	F
T	F	F	T	F	T	F
F	T	T	F	T	F	F
F	F	T	T	F	F	F

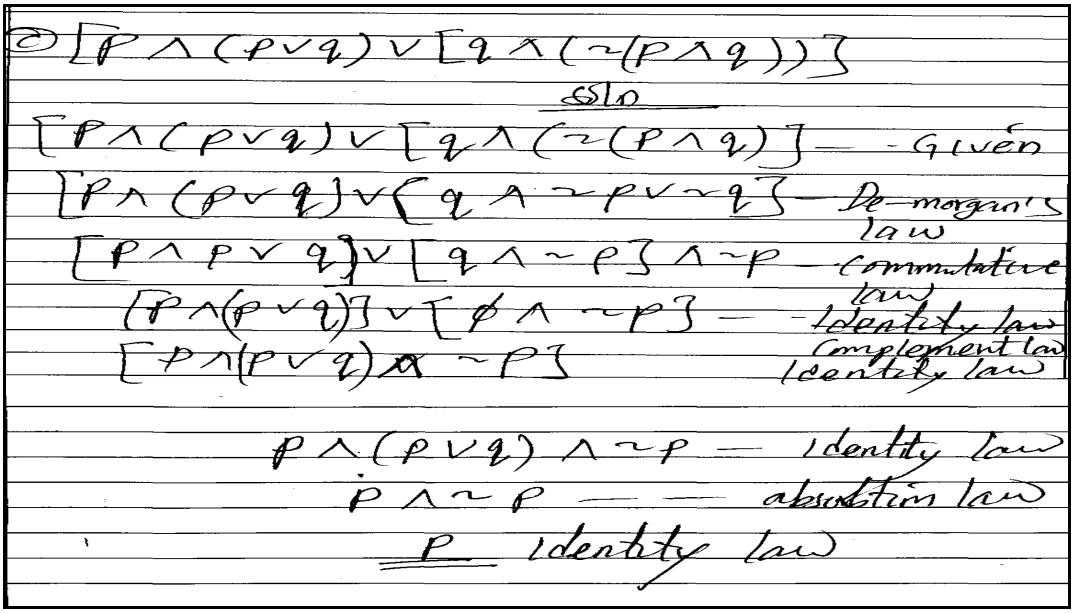
Again

for $(q \rightarrow p) \wedge (p \rightarrow q)$

Truth table

p	q	q → p (a)	p → q (b)	a ∧ b
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	F	F	F

∴ The statement $[(\sim p) \rightarrow (\sim q)] \wedge (p \rightarrow q)$ is equivalent to the statement $(q \rightarrow p) \wedge (p \rightarrow q)$ since both are contradiction statement or not taugtology statement.



Extract 12.2: A sample of incorrect responses to question 2

In Extract 12.2, in part (a), the candidate was unable to write the converse and inverse of the conditional statement correctly. In part (b), the candidate failed to provide the correct truth values and in part (c), the candidate was not able to use correctly laws of algebra of propositions to simplify logical compound statements.

2.2.3 Question 3: Vectors

The question comprised three parts (a), (b) and (c). In part (a), the candidates were required to find the work done by force $\underset{\sim}{F} = \underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ moving an object at a distance of 7m in the direction of the vector $\underset{\sim}{r} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$. Part (b), required the candidates to find a unit vector parallel to the displacement vector $\overrightarrow{PQ}$ if P and Q are points $P(3, -4, 6)$ and $Q(1, -3, 8)$ respectively. In part (c), the candidates were given the position vectors of points A and B as $\underline{a}$ and $\underline{b}$ respectively. If point C divides $\overrightarrow{AB}$ internally in the ratio of $2:1$, D divides $\overrightarrow{AB}$ externally in the ratio of $1:4$ and E divides $\overrightarrow{CD}$ internally in the ratio of $2:1$. The candidates were required to find the position vectors of C, D and E in terms of $\underline{a}$ and $\underline{b}$.

The data analysis shows that 5,182 (38.6%) candidates out of 13,443 (96.96%) who attempted the question scored 9 to 15 marks. A total of 4,479 (33.3%) candidates scored 5.5 to 8.5 marks and 3,782 (28.1%) candidates scored 0 to 5 marks. Figure 14 shows the percentage of candidates who obtained weak, average and good performance.

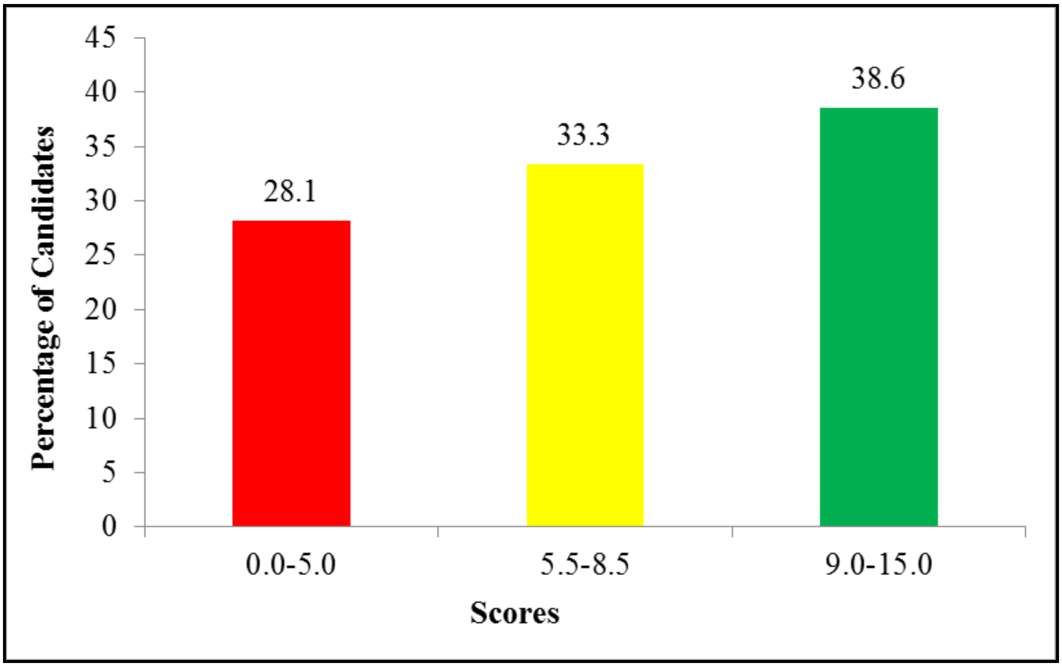


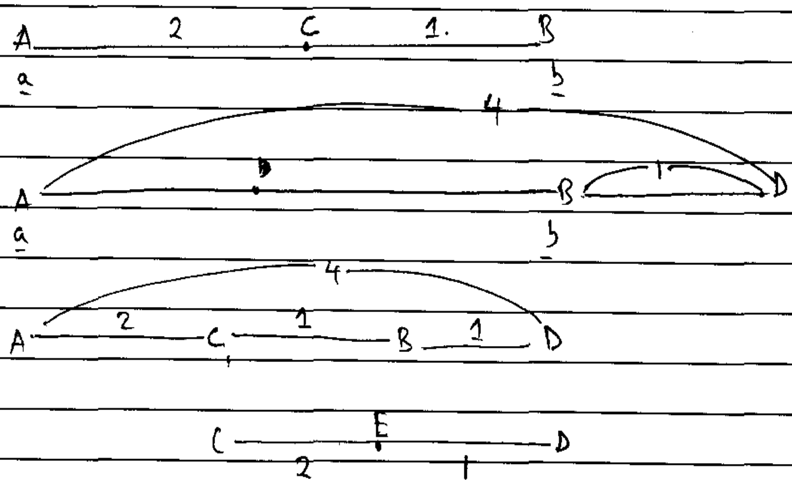
Figure 14: *Candidates' Performance in Question 3*

Figure 14 shows that most candidates (71.9%) scored from 5.5 to 15.0 marks, indicating that the candidates' performance in this question was good.

The candidates who attempted this question correctly were able to follow all the necessary steps of computation in each part. In part (a), they were able to find the displacement in the direction of the vector $\underset{\sim}{r} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$ using the formula $\underset{\sim}{d} = \frac{\underset{\sim}{r}}{|\underset{\sim}{r}|} \times 7$ to obtain $\underline{d} = \frac{21\underline{i} + 14\underline{j} + 28\underline{k}}{\sqrt{29}}$ correctly and finally, they were able to calculate the work done $W = \underline{F}.\underline{d}$ and obtained $\frac{77}{\sqrt{29}}$ or $\frac{77\sqrt{29}}{29}$ as a correct answer. In part (b), candidates were able to find the displacement vector $\overrightarrow{PQ}$ by subtracting position vector P from position vector Q and obtained $\overrightarrow{PQ} = -2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$. Finally, they computed the unit vector $\overrightarrow{PQ}$ using $\frac{\overrightarrow{PQ}}{|PQ|}$ to obtain $\frac{1}{3}\left(-2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}\right)$. In part (c), they were able to apply both internal and external ratio theorem step by step to obtain the position vectors $\underline{c}=\frac{1}{3}(\underline{a}+2\underline{b})$, $\underline{d}=\frac{1}{3}(4\underline{a}-\underline{b})$ and $\underline{e}=\frac{1}{3}(3\underline{d}+\underline{c})$. Finally, they used the obtained position vectors $\underline{c}$ and $\underline{d}$ into $\underline{e}=\frac{1}{3}(3\underline{d}+\underline{c})$ in order to get vector $\underline{e}$ in terms of $\underline{a}$ and $\underline{b}$. Then, they simplified to obtain $\underline{e}=\underline{a}$. Extract 3.1 illustrates the candidate who did this question correctly.

3a.	Workdone, $W = \underline{F} \cdot \underline{r}$	
	$\underline{F} = \underline{i} + 2\underline{j} + \underline{k}$	
	$\underline{r} = \dfrac{7(3\underline{i} + 2\underline{j} + 4\underline{k})}{\sqrt{(3)^2+(2)^2+(4)^2}}$	
	$\underline{r} = \dfrac{21}{\sqrt{29}}\underline{i} + \dfrac{14}{\sqrt{29}}\underline{j} + \dfrac{28}{\sqrt{29}}\underline{k}$	
	$W = (\underline{i} + 2\underline{j} + \underline{k}) \cdot \left(\dfrac{21}{\sqrt{29}}\underline{i} + \dfrac{14}{\sqrt{29}}\underline{j} + \dfrac{28}{\sqrt{29}}\underline{k}\right)$	
	$W = \dfrac{21}{\sqrt{29}} + \dfrac{28}{\sqrt{29}} + \dfrac{28}{\sqrt{29}}\ J$	
	$W = 77/\sqrt{29}\ J$	
	$W = 14.2985$ Joule	
	Workdone is 14.2985 Joule.	
b.	$\overrightarrow{PQ} = \begin{pmatrix} 1 \\ -3 \\ 8 \end{pmatrix} - \begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$	
	$\overrightarrow{PQ} = -2\underline{i} + \underline{j} + 2\underline{k}$	
	$\widehat{PQ} = \dfrac{-2\underline{i} + \underline{j} + 2\underline{k}}{\sqrt{(2)^2+(1)^2+(2)^2}}$	
	$\widehat{PQ} = -\frac{2}{3}\underline{i} + \frac{1}{3}\underline{j} + \frac{2}{3}\underline{k}$	

3 (c)



internal division

Case I

C divides $\overline{AB}$ internally

$m : n \quad 2 : 1.$

$$P = \left(\frac{m\underline{b}}{m+n} + \frac{n\underline{a}}{m+n} \right)$$

$$C = \left(\frac{2\underline{b}}{3} + \frac{\underline{a}}{3} \right)$$

$\therefore$ vector c is $\frac{2}{3}\underline{b} + \frac{1}{3}\underline{a}$

3	(c) Case II	
	Point D divides AB externally $m:n = 1:4$	
	$m = 1,\ n = 4$	
	$\cancel{D} = q = \left(\frac{m\underline{b}}{m-n} - \frac{n\underline{a}}{m-n}\right)$	
	$\underline{D} = \cancel{m}\left(\frac{\underline{b}}{1-4} - \frac{4\underline{a}}{1-4}\right)$	
	$\underline{D} = \left(\frac{-1}{3}\underline{b} + \frac{4}{3}\underline{a}\right)$	
	$\therefore$ vector $\underline{D} = \frac{-1}{3}\underline{b} + \frac{4}{3}\underline{a}$	
	Case III	
	E divides $\overline{CD}$ internally $m:n = 2:1$	
	$\underline{E} = \left(\frac{m\underline{d}}{m+n} + \frac{n\underline{c}}{m+n}\right)$	
	$\underline{E} = \left(\frac{2\left(-\frac{1}{3}\underline{b} + \frac{4}{3}\underline{a}\right)}{2+1} + \frac{1\left(\frac{2}{3}\underline{b} + \frac{1}{3}\underline{a}\right)}{2+1}\right)$	
	$\underline{E} = \frac{1}{3}\left[-\frac{2}{3}\underline{b} + \frac{8}{3}\underline{a}\right] + \frac{1}{3}\left[\frac{2}{3}\underline{b} + \frac{1}{3}\underline{a}\right]$	
	$\underline{E} = -\frac{2}{9}\underline{b} + \frac{8}{9}\underline{a} + \frac{2}{9}\underline{b} + \frac{1}{9}\underline{a}$	
	$\underline{E} = 0\underline{b} + \underline{a}$	
	$\therefore$ vector $\underline{E} = 0\underline{b} + \underline{a}$ or $\underline{E} = \underline{a}$	

Extract 13.1: A sample of correct responses to question 3

In Extract 13.1, the candidate had enough skills that enabled him/her to calculate the work done properly, unit vector and applied the ratio theorem to get the required position vectors.

Apart from good performance on this question, there were candidates who did not do this question well. In part (a), some candidates multiplied the distance and direction vectors as $7(3i+2j+4k)$ and then they dotted it to the force $i+2j+k$ to obtain 77 Joules contrary to $\frac{77}{\sqrt{29}}$ Joules. Again, it was noted that others calculated directly the work done by multiplying force by distance, that is $\text{work done} = (i+2j+k)\times 7$. It was further noted that others calculated the unit vector of a force as $\hat{F} = \frac{i+2j+k}{\sqrt{1^2+2^2+1^2}} = \frac{i+2j+k}{\sqrt{6}}$ and finally dotted the result with the product of distance and directional vector to obtain $\frac{i+2j+k}{\sqrt{6}}.7(3i+2j+4k)$. Furthermore, it was observed that a few candidates who followed all steps correctly obtained $F=(1+2i+k).\left(\frac{21i+14j+28k}{\sqrt{29}}\right)$ but were not able to realize that the dot product results into scalar, then left the answer as $\left(\frac{21i+28j+28k}{\sqrt{29}}\right)$.

Not only that but also in part (b), some candidates just crossed vectors P and Q, that is $P\times Q = 14i+18j-5k$. Then, they calculated its unit vector as $\overline{PQ} = \frac{14i+18j-5k}{\sqrt{545}}$ which was completely incorrect. Again, it was observed that some candidates lacked knowledge of integer operation. For example, they wrote displacement $\overrightarrow{PQ}$ as $(1-3)i+(-3-4)j+(8-6)k$ instead of $(1-3)i+(-3--4)j+(8-6)k$ that led to incorrect unit vector $\frac{-2i-7j+2k}{\sqrt{65}}$. Further observations showed that some of the candidates were using the concept of projection on vectors, that is, $\text{Proj} = \frac{(3i-4j+6k).(i-3j+8k)}{\sqrt{1^2+3^2+8^2}}$ and treated it as the unit vector.

In part (c), a few candidates found position vectors for $\underline{c}$ and $\underline{d}$ correctly as $\underline{c} = \frac{1}{3}(\underline{a}+2\underline{b})$ and $\underline{d} = \frac{1}{3}(4\underline{a}-\underline{b})$. However, they did not realize that vector E divides C and D externally, then they wrote $e = \frac{2b+a}{3}$ instead of $e = \frac{2d+c}{3}$. It was further noted that other candidates reversed the position vectors to be applied in the ratio theorem as

$\underset{\sim}{c} = \dfrac{ma+nb}{m+n}$ and then they substituted the ratio $m:n = 2:1$ to obtain $\underset{\sim}{c} = \dfrac{2a+b}{3}$.

Extract 13.2 shows a sample of incorrect responses from one of the candidates.

3a From

Workdone = Force X Distance

Force = $i + 2j + k$

Distance = 7

Then

Workdone = F X D

$= (i + 2j + k)\,7$

$= 7i + 14j + 7k$

W.D. $= 7i + 14j + 7k$

$\therefore$ Workdone is $7i + 14j + 7k$.

3b $P\phi = (3, -4, 6)$

$\phi = (1, -3, 8)$

So

$P\phi = 2i, -j - 2k$

let a = Vector $P\phi$

So vector parallel to a make an angle of 0

let that vector to be b

from

$(\cos\phi)\,|a||b| = a \cdot b$

$\cos\phi = \dfrac{a \cdot b}{|a||b|}$

$\cos 0 = \dfrac{b\,(2i, -j - 2k)}{9}$

$1 = \dfrac{b\,(2i - j - 2k}{9}$

$(2i - j - 2k)\,b = 9$

$b =$

3b $b = \dfrac{9}{2i - j - 2k}$

$b = 9\left(\dfrac{1}{2i - j - 2k}\right)$

$\therefore$ The vector parallel to the displacement $P\phi$ is

$9\left(\dfrac{1}{2i - j - 2k}\right)$

3C From

Division of line Vector

let

$a = a_1 i + a_2 j + a_3 k$

$b = b_1 i + b_2 j + b_3 k$

Internal division of line

$(x, y, z) = \left(\frac{M}{M+n}\right)b + \left(\frac{n}{M+n}\right)a$

$M : n = 2 : 1$

$= \frac{2}{3}(b_1 i + b_2 j + b_3 k) + \frac{1}{3}(a_1 i + a_2 j + a_3 k)$

$\frac{1}{3}(2b_1 i + 2b_2 j + 2b_3 k + a_1 i + a_2 j + a_3 k)$

$\frac{1}{3}((2b_1 + a_1)i + (2b_2 + a_2)j + (2b_3 + a_3)k)$

$\therefore$ position vector of C is $\frac{1}{3}((2b_1 + a_1)i + (2b_2 + a_2)j + (2b_3 + a_3)k)$

3C For external division

$\left(\frac{m}{M-n}\right)b - \left(\frac{n}{M-n}\right)a$

But

$M : n = 1 : 4$

$= \frac{1}{-3}(b_1 i + b_2 j + b_3 k) - \frac{4}{-3}(a_1 i + a_2 j + a_3 k)$

$\frac{1}{-3}((b_1 - 4a_1)i + (b_2 - 4a_2)j + (b_3 - 4a_3)k)$

$\therefore$ position vector d is $-\frac{1}{3}((b_1 - 4a_1)i + (b_2 - 4a_2)j - (b_3 - 4a_3)k)$

For Internally division

$\left(\frac{m}{m+n}\right)b + \left(\frac{n}{M+n}\right)a$

But

$M : n = 2 : 1$

$\frac{2}{3}(b_1 i + b_2 j + b_3 k) + \frac{1}{3}(a_1 i + a_2 j + a_3 k)$

$\frac{1}{3}((2b_1 + a_1)i + (2b_2 + a_2)j + (2b_3 + a_3)k)$

$\therefore$ The position vector of vector E is

$\frac{1}{3}((2b_1 + a_1)i + (2b_2 + a_2)j + (2b_3 + a_3)k)$

Hence shown

Extract 13.2: A sample of incorrect responses to question 3

In Extract 13.2, the candidate failed to calculate the work done and the unit vector in the direction of $\overrightarrow{PQ}$ and also failed to apply both external and internal ratio theorems.

2.2.4 Question 4: Complex Numbers

This question had four parts, (a), (b), (c) and (d). In part (a), the candidates were required to express $\sqrt{1+i}$ in polar form. In part (b), they were required to use the results of part (a) to show that $\tan\frac{\pi}{8}=\sqrt{2}-1$. In part (c), they were required to prove that $Arg\left(\frac{z_1}{z_2}\right)=Arg\left(z_1\right)-Arg\left(z_2\right)$ if $z_1=r_1\left(\cos\theta_1+i\sin\theta_1\right)$ and $z_2=r_2\left(\cos\theta_2+i\sin\theta_2\right)$. In part (d), the candidates were given the complex numbers $z_1=\frac{c}{1+i}$ and $z_2=\frac{d}{1+2i}$ where $c,d\in\mathbb{R}$. Then, they were asked to find the values of c and d such that $z_1+z_2=1$.

The analysis shows that 36.4 per cent of the candidates who attempted this question scored from 0 to 5 marks, 41.9 per cent scored from 5.5 to 8.5 marks and 21.7 per cent scored from 9 to 15 marks. Generally, the candidates' performance in this question was good as 63.6 per cent of the candidates got more than 5 marks. Figure 15 illustrates the candidates' performance in this question.

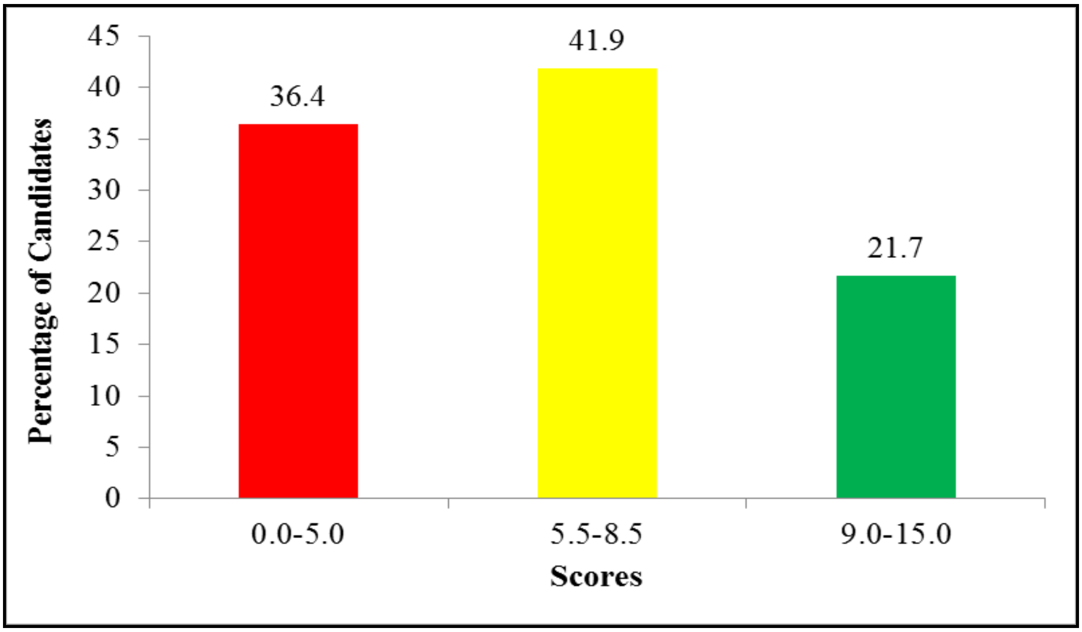


Figure 15: *Candidates' Performance in Question 4*

Further, the analysis revealed that those who had high performance had good understanding of the topic and approached the question according to what was asked. In part (a), the candidates were able to express the complex number $1+i$ in polar form and then used De-Moivre's theorem to obtain $\sqrt{1+i}=\left(\sqrt{2}\right)^{\frac{1}{2}}\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)$. In part (b), the candidates managed to use the result of part (a) to show that $\tan\frac{\pi}{8}=\sqrt{2}-1$. They squared both sides of the equation $\sqrt{1+i}=\left(\sqrt{2}\right)^{\frac{1}{2}}\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)$ to obtain $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i=\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}$ and from this they deduced that $\tan\left(\frac{\pi}{4}\right)=1$. They used the identity $\tan\theta=\dfrac{2\tan\left(\frac{\theta}{2}\right)}{1-\tan^2\left(\frac{\theta}{2}\right)}$ to obtain $\tan\left(\frac{\pi}{4}\right)=\dfrac{2\tan\left(\frac{\pi}{8}\right)}{1-\tan^2\left(\frac{\pi}{8}\right)}$. Then, they formulated the quadratic equation in $\tan\left(\frac{\pi}{8}\right)$, that is, $\tan^2\left(\frac{\pi}{8}\right)+2\tan\left(\frac{\pi}{8}\right)-1=0$, and on solving it, they obtained $\tan\left(\frac{\pi}{8}\right)=\sqrt{2}-1$.

In part (c), the candidates proved $Arg\left(\frac{z_1}{z_2}\right)=Arg\left(z_1\right)-Arg\left(z_2\right)$ using the concept of operation of complex numbers and the knowledge of the compound angle formulae. First, they computed $\frac{z_1}{z_2}$ from the given expressions $z_1=r_1\left(\cos\theta_1+i\sin\theta_1\right)$ and $z_2=r_2\left(\cos\theta_2+i\sin\theta_2\right)$ then obtained $\frac{z_1}{z_2}=\frac{r_1}{r_2}\left(\cos\left(\theta_1-\theta_2\right)+i\sin\left(\theta_1-\theta_2\right)\right)$. Thus, $Arg\left(\frac{z_1}{z_2}\right)=\theta_1-\theta_2$ and from this they deduced that $Arg\left(\frac{z_1}{z_2}\right)=Arg\left(z_1\right)-Arg\left(z_2\right)$. In part (d), the candidates applied the concept of operation on the equation $z_1+z_2=1$ to obtain $\frac{c(1+i)}{2}+\frac{d(1-2i)}{5}=1$. From this equation, they formulated two simultaneous equations in *c* and *d*, $5c+2d=10$ and $5c+4d=0$ by comparing real and imaginary parts. On solving the equations, they obtained the required values of $c=4$ and $d=-5$. Extract 14.1 shows a sample of responses from one of the candidates who correctly answered this question.

Q4. (a) let z ·

$a = \sqrt{1+i}$.

$a^2 = 1 + i$

$r = \sqrt{x^2+y^2}$

$r = \sqrt{1^2+1^2}$

$r = \sqrt{2}$.

$\arg(c) = \tan^{-1}(y/x)$

$= \tan^{-1}(1/1)$

$\theta = \pi/4$

then $a^2 = \sqrt{2}\,(\cos \pi/4 + i \sin \pi/4)$ from $z = r\cos\theta + ir\sin\theta$.

$a = (\sqrt{2})^{1/2}\,(\cos \pi/4 + i\sin \pi/4)^{1/2}$.

$a = 2^{1/4}\,(\cos \pi/8 + i \sin \pi/8)$ ·

since $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Now $a = 2^{1/4}\,(\cos \pi/8 + i\sin \pi/8)$

$\therefore \sqrt{1+i} = 2^{1/4}\,(\cos \pi/8 + i \sin \pi/8)$

$\therefore \sqrt{1+i} = 2^{1/4}\,(\cos \pi/8 + i \sin \pi/8)$

(b) Required to show $\tan \pi/8 = \sqrt{2} - 1$

From.

$1 + i = \sqrt{2}\cos \pi/4 + (\sqrt{2}\sin \pi/4)i$

$1 = \sqrt{2}\cos \pi/4$ on comparing real and imaginary parts.

$1 = \sqrt{2}\sin \pi/4$.

Divide them

$\tan \pi/4 = 1$.

From $\tan \pi/4 = \dfrac{2\tan \pi/8}{1-\tan^2 \pi/8}$ since $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

but $\tan \pi/4 = 1$.

$2\tan \pi/8 = 1 - \tan^2 \pi/8$

$\tan^2 \pi/8 + 2\tan \pi/8 - 1 = 0$

04 (b) ~~tan~~ From Equation

$a = 1 \quad b = 2, \quad c = -1.$

From $\tan \pi/8 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\tan \pi/8 = \dfrac{-2 \pm \sqrt{2^2 + 4}}{2}$

$\tan \pi/8 = \dfrac{-2 \pm \sqrt{8}}{2}.$

$\tan \pi/8 = \dfrac{-2 \pm 2\sqrt{2}}{2}.$

$\tan \pi/8 = -1 \pm \sqrt{2}.$

Ignore -ve part. (negative part)

$\tan \pi/8 = -1 + \sqrt{2}.$

$\therefore \tan \pi/8 = \sqrt{2} - 1$

Hence shown.

(c). Given.

$Z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$

$Z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$

Required to prove that

$\arg\left(\dfrac{Z_1}{Z_2}\right) = \arg(Z_1) - \arg(Z_2).$

Consider left side.

$\dfrac{Z_1}{Z_2} = \dfrac{r_1 \cos \theta_1 + i\sin \theta_1}{r_2 \cos \theta_2 + i\sin \theta_2}$

$\dfrac{Z_1}{Z_2} = \dfrac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)\right)$

$\arg\left(\dfrac{Z_1}{Z_2}\right) = \theta_1 - \theta_2.$

From $Z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$

$\arg(Z_1) = \theta_1$

04 (b) ~~tan~~ From Equation

$a = 1 \quad b = 2, \quad c = -1.$

From $\tan \pi/8 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\tan \pi/8 = \dfrac{-2 \pm \sqrt{2^2 + 4}}{2}$

$\tan \pi/8 = \dfrac{-2 \pm \sqrt{8}}{2}.$

$\tan \pi/8 = \dfrac{-2 \pm 2\sqrt{2}}{2}$

$\tan \pi/8 = -1 \pm \sqrt{2}.$

Ignore -ve part. (negative part)

$\tan \pi/8 = -1 + \sqrt{2}.$

$\therefore \tan \pi/8 = \sqrt{2} - 1$

Hence shown.

(c). Given.

$z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$

$z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$

Required to prove that

$\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2).$

Consider left side.

$z_1/z_2 = \dfrac{r_1 \cos \theta_1 + i \sin \theta_1}{r_2 \cos \theta_2 + i \sin \theta_2}$

$\dfrac{z_1}{z_2} = r_1/r_2 \left(\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)\right)$

$\arg\left(\dfrac{z_1}{z_2}\right) = \theta_1 - \theta_2.$

From $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$

$\arg(z_1) = \theta_1$

04 (c). Also $z_2 = r_2(\cos \theta_2 + i \sin \theta_2).$

$\arg(z_2) = \theta_2.$

$\arg(z_1) - \arg(z_2) = \theta_1 - \theta_2.$

Since $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2) = \theta_1 - \theta_2$

Hence proved.

(a) Given. $z_1 = \dfrac{c}{1+i}$ and $z_2 = \dfrac{d}{1+2i}$

$z_1 + z_2 = 1$.

$\dfrac{c}{1+i} + \dfrac{d}{1+2i} = 1$.

$\dfrac{c(1+2i) + d(1+i)}{(1+i)(1+2i)} = 1$.

$c + 2ci + d + di = (1+i)(1+2i)$

$c+d + (2c+d)i = 1 + 2i + i + 2i^2$

$(c+d) + (2c+d)i = 1 + 2i + i - 2$

$(c+d) + (2c+d)i = -1 + 3i$

On comparing real and imaginary parts.

$c + d = -1$

$2c + d = 3$

Solving the equations simultaneously.

$c = 4, \; d = -5$

$\therefore c = 4$ and $d = -5$

Extract 14.1: A sample of correct responses to question 4

In Extract 14.1, the candidate had enough knowledge and skills on the topic about complex numbers which made him/her follow all the necessary steps in answering each part of the question correctly.

On the other hand, the analysis shows that there were candidates who performed this question poorly due to different challenges they faced. In part (a), some candidates confused polar form with polynomial form. They expressed $\sqrt{1+i}$ in the form $x+iy$ by setting $\sqrt{1+i} = x+iy$. After solving for x and y, they wrote $\sqrt{1+i} = \pm\left(\dfrac{\sqrt{1+\sqrt{2}}}{2} + \dfrac{\sqrt{2}}{2\sqrt{1+\sqrt{2}}}i\right)$. Others wrote $\cos\theta + i\sin\theta = \sqrt{1+i}$ and compared real and imaginary parts, which was still a wrong approach to this question. There were other candidates who misinterpreted the question. They applied De-Moivre's theorem to express $(x+i)^2$ in polar form instead of $\sqrt{1+i}$. Also, there were some candidates who applied De-Moivre's theorem to calculate the square roots of $(1+i)$ instead expressing $\sqrt{1+i}$ in its polar form. They wrote

$\sqrt{1+i} = \left(\sqrt{2}\right)^{\frac{1}{2}}\left(\cos\left(\dfrac{\frac{\pi}{4}+2\pi n}{2}\right)+i\sin\left(\dfrac{\frac{\pi}{4}+2\pi n}{2}\right)\right)$ and on setting $n=0$ and $n=1$. They obtained $\sqrt{1+i} = \left(\sqrt{2}\right)^{\frac{1}{2}}\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)$, $\left(\sqrt{2}\right)^{\frac{1}{2}}\left(\cos\frac{9\pi}{8}+i\sin\frac{9\pi}{8}\right)$. In part (b), some candidates failed to separate real and imaginary parts from the equation $\sqrt{1+i} = \left(\sqrt{2}\right)^{\frac{1}{2}}\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)$, which is the technique they could use to deduce the values of $\cos\frac{\pi}{4}$ and $\sin\frac{\pi}{4}$ then $\tan\frac{\pi}{8}$. They expressed $\cos\frac{\pi}{4}$ and $\sin\frac{\pi}{4}$ the subjects to obtain $\cos\left(\frac{\pi}{4}\right) = \dfrac{\sqrt{1+i}-i\left(\sqrt{2}\right)^{\frac{1}{2}}\sin\left(\frac{\pi}{4}\right)}{\left(\sqrt{2}\right)^{\frac{1}{2}}}$ and $\sin\left(\frac{\pi}{4}\right) = \dfrac{\sqrt{1+i}-\left(\sqrt{2}\right)^{\frac{1}{2}}\cos\left(\frac{\pi}{4}\right)}{i\left(\sqrt{2}\right)^{\frac{1}{2}}}$. From these equations, they deduced that $\tan\left(\frac{\pi}{4}\right) = \dfrac{\left(\sqrt{1+i}-\left(\sqrt{2}\right)^{\frac{1}{2}}\cos\left(\frac{\pi}{4}\right)\right)\left(\sqrt{2}\right)^{\frac{1}{2}}}{\left(\sqrt{2}\right)^{\frac{1}{2}}i\left(\sqrt{1+i}-i\left(\sqrt{2}\right)^{\frac{1}{2}}\sin\left(\frac{\pi}{4}\right)\right)}$ which was not correct final answer.

In part (c), the candidates applied a wrong approach in trying to prove the given statement. They expressed the right hand side of the statement as $\arg(z_1)-\arg(z_2) = \tan^{-1}\left(\dfrac{r_1\sin\theta_1}{r_1\cos\theta_1}\right)-\tan^{-1}\left(\dfrac{r_2\sin\theta_2}{r_2\cos\theta_2}\right) = \tan^{-1}(\tan\theta_1-\tan\theta_2)$. This was not only a wrong approach but also an incorrect concept of inverse of trigonometric functions. Furthermore, there were candidates who failed to write the correct expression for argument and modulus for a complex number. They wrote Argument $(r) = \sqrt{x^2-y^2}$ and $r = \left(\dfrac{\cos\theta}{z}+\dfrac{i\sin\theta}{z}\right)$.

In part (d), the candidates failed to apply the basic operation of complex numbers. They expressed z_1 and z_2 in polar form and finally they wrote $z_1+z_2 = \dfrac{c}{\sqrt{2}\left(\cos 45^\circ+i\sin 45^\circ\right)}+\dfrac{d}{\sqrt{2}\left(\cos 63.43^\circ+i\sin 63.43^\circ\right)} = 1$. Others did a wrong substitution of z_1 and z_2 into $z_1 + z_2 = 1$ to obtain $c + ci + d + \frac{di}{2} = 1 + 0i$. This led to wrong answers of *c* and *d* as -1 and 2 respectively. Futhermore, some candidates obtained the equation $c(1+2i) + d(1+i) = (1+i)(1+2i)$ and failed to make a correct comparison. They wrote $c(1+2i) = (1+i)(1+2i)$ and $d(1+i) = (1+i)(1+2i)$. Extract 14.2 provides a sample of responses from one of the candidates who used wrong concepts in solving the question.

4(a) Soln

Given:

$\sqrt{1+i}$

Required to express into polar form.

Let $\sqrt{1+i}$ be Z.

$Z = \sqrt{1+i}$

from,

$Z = \cos\theta + i\sin\theta$

$(\sqrt{(1+i)})^2 = (\cos\theta + i\sin\theta)^2$

$1 + i = (\cos\theta + i\sin\theta)^2$

$1 + i = (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$

$1 + i = (\cos^2\theta + i\sin\theta\cos\theta + i\sin\theta + \cos\theta + i^2\sin^2\theta$

$1 + i = \cos^2\theta + 2i\sin\theta\cos\theta + i^2\sin^2\theta$

$1 + i = \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta$

$1 + i = \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta$

Now comparing the terms

$\cos^2\theta - \sin^2\theta = 1$ - - - - - (i)

$2i\sin\theta\cos\theta = i$ - - - - - (ii)

$2\sin\theta\cos\theta = 1$ - - - - - (iii)

Put eqn (iii) into eqn (i)

$\cos^2\theta - \sin^2\theta = 2\sin\theta\cos\theta$

$\cos^2\theta - \sin^2\theta - 2\sin\theta\cos\theta = 0$

(c) Soln

Given: $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Required to prove that:

$\operatorname{Arg}\left(\frac{Z_1}{Z_2}\right) = \operatorname{Arg}(Z_1) - \operatorname{Arg}(Z_2)$.

from the given data above;

$\frac{Z_1}{Z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$

4(c)	But form,
	$Z = r(\cos\theta + i\sin\theta)$
	$\text{Argument}(r) = \sqrt{x^2 + y^2}$
	$r = \left(\frac{\cos\theta}{Z} + \frac{i\sin\theta}{Z}\right)$
	$\lvert r \rvert = \sqrt{\frac{\cos^2\theta}{Z^2} + \frac{i^2\sin^2\theta}{Z^2}}$
	$\lvert r \rvert = \sqrt{\frac{\cos^2\theta}{Z^2} - \frac{\sin^2\theta}{Z^2}}$
	but $\cos^2\theta - \sin^2\theta = 1$.
	Now,
	$r = \text{Arg}(Z)$
	so for the data above.
	$\frac{Z_1}{Z_2} = \frac{\cos\theta_1 - i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2}$.
	$\text{Arg}\left[\frac{Z_1}{Z_2}\right] = \text{Arg}(Z_1) - \text{Argum}(Z_2)$
	$\therefore \text{Arg}\left(Z_1/Z_2\right) = \text{Arg}(Z_1) - \text{Arg}(Z_2)$
	hence proved.

Extract 14.2: A sample of incorrect responses to question 4(a) and 4(c)

In Exact 14.2, the candidate failed to change the given expression into polar form in part (a) and also in part (c), the candidate failed to prove the argument of the ratio of complex number being equal to the difference of argument of the ratios.

2.2.5 Question 5: Trigonometry

The question consisted of parts (a), (b), (c) and (d). In part (a), the candidates were required to show that $\frac{\sin 3\alpha}{\sin\alpha} - \frac{\cos 3\alpha}{\cos\alpha} = 2$. In part (b), the candidates were required to prove that $\frac{\sin x + \sin 2x + \sin 3x}{\cos x + \cos 2x + \cos 3x} = \tan 2x$. In part (c), they were required to solve for β in the equation $\tan^{-1}\left(\frac{\beta - 1}{\beta - 2}\right) + \tan^{-1}\left(\frac{\beta + 1}{\beta + 2}\right) = \frac{\pi}{4}$. In part (d), they were required to re-write $4\cos\theta + 3\sin\theta$ in the form $R\cos(\theta - \alpha)$, and then to solve the equation $4\cos\theta + 3\sin\theta = \frac{5\sqrt{2}}{2}$ in the interval $\frac{\pi}{2} \le \theta \le 2\pi$.

The analysis of data shows that the question was attempted by 10,331 (76.8%) candidates, out of whom 78.3 per cent of the candidates scored 12 to 20 marks. Further analysis shows that 13.8 per cent of the candidates scored 7 to 11.5 marks and 8.0 per cent scored from 0 to 6.5 marks. The data also shows that 1153 (11.2%) candidates scored 20 marks while 125 (1.2%) candidates scored 0. Therefore, the candidates' performance in this question was good. The summary of candidates' performance is presented in Figure 16.

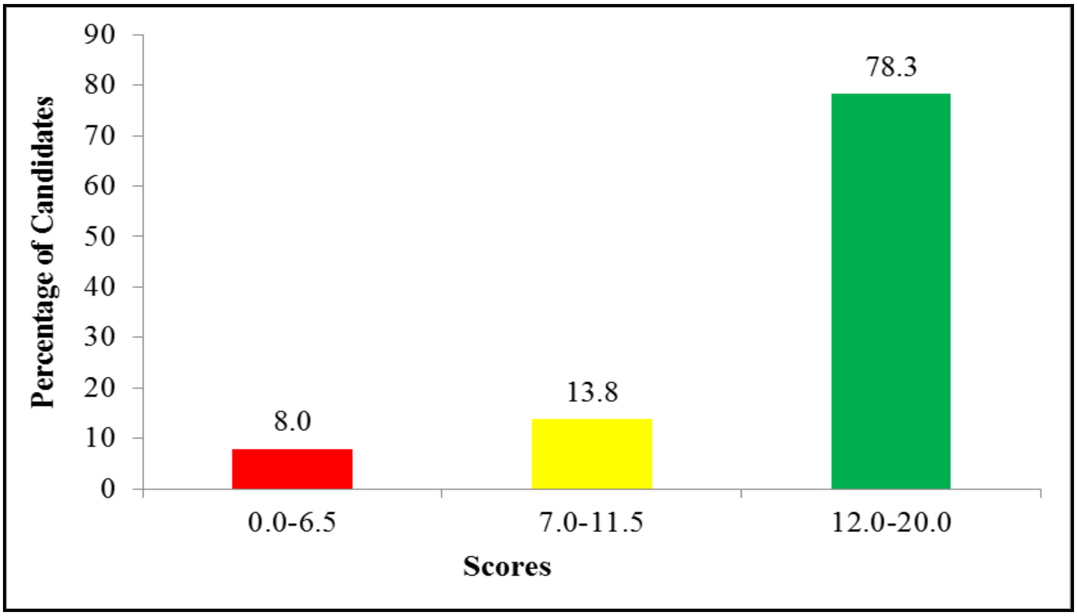


Figure 16: *Candidates' Performance in Question 5*

The candidates who performed well in this question had the appropriate knowledge of the topic on trigonometry and followed all the necessary steps. In part (a), the candidates applied the triple angle formulae, $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$ and $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$ then substituted these expressions into $\frac{\sin 3\alpha}{\sin\alpha} - \frac{\cos 3\alpha}{\cos\alpha}$, and on simplifying, they obtained 2. Other candidates combined $\frac{\sin 3\alpha}{\sin\alpha} - \frac{\cos 3\alpha}{\cos\alpha}$ with $\frac{\sin 3\alpha\cos\alpha - \cos 3\alpha\sin\alpha}{\sin\alpha\cos\alpha}$, and from this combination, some candidates applied the concept of compound angle formulae $\sin 3\alpha\cos\alpha - \cos 3\alpha\sin\alpha = \sin(3\alpha - \alpha)$ while others applied the concept of factor formulae

$\sin 3\alpha \cos\alpha - \cos 3\alpha \sin\alpha = \frac{1}{2}(\sin 4\alpha + \sin 2\alpha) - \frac{1}{2}(\sin 4\alpha - \sin 2\alpha)$. On simplifying, they all ended up with the required result. In part (b), the candidates applied the concept of factor formulae $\sin x + \sin 2x + \sin 3x = \sin 2x + (\sin x + \sin 3x) = \sin 2x + 2\sin\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right)$ and $\cos x + \cos 2x + \cos 3x = \cos 2x + (\cos x + \cos 3x) = \cos 2x + 2\cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right)$.

Then they substituted these expressions into $\frac{\sin x + \sin 2x + \sin 3x}{\cos x + \cos 2x + \cos 3x}$ and simplified it to obtain $\tan 2x$.

In part (c), the candidate applied the concepts of inverse of trigonometric functions as well as the concept of compound angle formulae to obtain the required value of β. That is, they considered $\tan^{-1}\left(\frac{\beta-1}{\beta-2}\right) = \text{A}$ then $\Rightarrow \tan A = \frac{\beta-1}{\beta-2}$ and $\tan^{-1}\left(\frac{\beta+1}{\beta+2}\right) = B$ then $\Rightarrow \tan B = \frac{\beta+1}{\beta+2}$. Thereafter, they applied the tangent of $A+B = \frac{\pi}{4}$ to obtain the equation $\frac{\beta-1}{\beta-2} + \frac{\beta+1}{\beta+2} = 1 - \left(\frac{\beta-1}{\beta-2}\right)\left(\frac{\beta+1}{\beta+2}\right)$. Then, they simplified and solved this equation to obtain the required value β as $\pm\frac{\sqrt{2}}{2}$.

In part (d), the candidates were able to express $4\cos\theta + 3\sin\theta$ in the form $R\cos(\theta-\alpha)$ by applying the concept of compound angle formulae. They expanded $R\cos(\theta-\alpha)$ into $R(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$ and equated this expression to $4\cos\theta + 3\sin\theta$. On comparing, they obtained $R\cos\alpha = 4$ and $R\sin\alpha = 3$. Then, they determined $\tan\alpha = \frac{3}{4}$ which led to obtain $\alpha = \tan^{-1}\left(\frac{3}{4}\right)$ and $R = 5$. Using these results, they solved the equation $4\cos\theta + 3\sin\theta = \frac{5\sqrt{2}}{2}$ by replacing $4\cos\theta + 3\sin\theta$ with $5\cos\left(\theta - \tan^{-1}\left(\frac{3}{4}\right)\right)$ and finally they obtained the required values of θ, 1.43 rad (or 81.87°) and 6.14 rad (or 351.87°).

Extract 15.1 shows a sample of the correct solutions from one of the candidates who had sufficient knowledge on this topic.

5(a) solution

Required to show that for all values of α

$$\frac{\sin 3\alpha}{\sin \alpha} - \frac{\cos 3\alpha}{\cos \alpha} = 2$$

Consider left hand side

$$\frac{\sin 3\alpha}{\sin \alpha} - \frac{\cos 3\alpha}{\cos 3\alpha} = \frac{\sin 3\alpha \cos \alpha - \cos 3\alpha \sin \alpha}{\sin \alpha \cos \alpha}.$$

$$= \frac{\sin 3\alpha \cos \alpha - \cos 3\alpha \sin \alpha}{\sin \alpha \cos \alpha}.$$

$$= \frac{\sin (3\alpha - \alpha)}{\sin \alpha \cos \alpha}$$

$$= \frac{\sin 2\alpha}{\cos \alpha \sin \alpha}.$$

$$= \frac{2 \sin \alpha \cos x}{\sin \alpha \cos x}.$$

$$= 2$$

Since left hand side = Right hand side

Therefore

$$\frac{\sin 3\alpha}{\sin \alpha} - \frac{\cos 3\alpha}{\cos \alpha} = 2$$

Hence shown!

solution

Required to prove that $\frac{\sin X + \sin 2X + \sin 3X}{\cos X + \cos 2X + \cos 3X} = \tan 2X$

Consider left hand side.

$$\frac{\sin X + \sin 2X + \sin 3X}{\cos X + \cos 2X + \cos 3X} = \frac{\sin X + \sin 3X + \sin 2X}{\cos X + \cos 3X + \cos 2X}$$

$$= \frac{2 \sin 2X \cos X + \sin 2X}{2 \cos 2X \cos X + \cos 2X}$$

$$= \frac{2 \sin 2X (\cos X + 1)}{2 \cos 2X (\cos X + 1)}$$

$$= \tan 2X$$

Since left hand side = Right hand side.

Therefore

$$\frac{\sin X + \sin 2X + \sin 3X}{\cos X + \cos 2X + \cos 3X} = \tan 2X$$

Hence proved.

5(c) Solution

Required to solve for value of B.

Given

$$\tan^{-1}\left(\frac{B-1}{B-2}\right) + \tan^{-1}\left(\frac{B+1}{B+2}\right) = \pi/4$$

let $A = \tan^{-1}\left(\frac{B-1}{B-2}\right)$ $B = \tan^{-1}\left(\frac{B+1}{B+2}\right)$

then

$A + B = \pi/4$ Apply tan both side

$$\tan(A+B) = \tan \pi/4$$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \tan \pi/4$$

but $\tan \pi/4 = 1$

$$\tan A + \tan B = 1 - \tan A \tan B$$

$$\tan\left(\tan^{-1}\left(\frac{B-1}{B-2}\right)\right) + \tan\left(\tan^{-1}\left(\frac{B+1}{B+2}\right)\right) = 1 - \tan\left(\tan^{-1}\left(\frac{B-1}{B-2}\right)\right)\tan\left(\tan^{-1}\left(\frac{B+1}{B+2}\right)\right)$$

$$\frac{B-1}{B-2} + \frac{B+1}{B+2} = 1 - \left(\frac{B-1}{B-2}\right)\left(\frac{B+1}{B+2}\right)$$

$$(B-1)(B+2) + (B-2)(B+1) = (B-2)(B+2) - (B-1)(B+1)$$

$$B^2 + \cancel{2B} - B - 2 + B^2 + B - \cancel{2B} - 2 = B^2 - 4 - B^2 + 1$$

$$2B^2 - 4 = -4 + 1$$

$$2B^2 - 4 = -3$$

$$2B^2 = -3 + 4$$

$$\frac{2B^2}{2} = \frac{1}{2}$$

$$\sqrt{B^2} = \sqrt{1/2}$$

$$B = \sqrt{1/2}$$

$\therefore$ the value of B is $\sqrt{1/2}$

i)

solution:

Given: $4\cos\theta + 3\sin\theta$

Required: to express in form $R\cos(\theta - \alpha)$

let

$R\cos(\theta - \alpha) = 4\cos\theta + 3\sin\theta$

$R\cos\theta\cos\alpha + R\sin\theta\sin\alpha = 4\cos\theta + 3\sin\theta$.

$R\cos\theta\cos\alpha = 4\cos\theta$

$R\cos\alpha = 4 \quad ---\ i)$

$R\sin\theta\sin\alpha = 3\sin\theta$

$R\sin\alpha = 3 \quad ---\ ii)$

square eqn i) and eqn ii)

$R^2\cos^2\alpha = 16.$

$R^2\sin^2\alpha = 9$

then adding the squared equations

$R^2(\cos^2\alpha + \sin^2\alpha) = 16 + 9$

$\sqrt{R^2} = \sqrt{25}$

$R = 5$

then dividing eqn ii) and eqn i)

$\frac{R\sin\alpha}{R\cos\alpha} = 3/4$

$\tan\alpha = 3/4$

$\alpha = \tan^{-1}(3/4)$

$\alpha = 36.9°$

Now

$4\cos\theta + 3\sin\theta = 5\cos(\theta - 36.9°)$

$\therefore \quad 4\cos\theta + 3\sin\theta = 5\cos(\theta - 36.9°)$

Required to solve for.

$4\cos\theta + 3\sin\theta = 5\sqrt{2}$

$\overline{2}$

since $\cdot\ 4\cos\theta + 3\sin\theta = 5\cos(\theta - 36.9°)$

$$\frac{5\cos(\theta - 36.9^\circ)}{5} = \frac{5\sqrt{2}}{2}$$

$$\cos(\theta - 36.9^\circ) = \frac{\sqrt{2}}{2}$$

$$\theta - 36.9^\circ = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$\theta - 36.9^\circ = 45^\circ = \pi/4$$

from the general solution of cosine

$$\theta = 360n \pm \alpha$$

$$\theta = 2\pi n \pm \alpha$$

$\theta - 36.9^\circ = 2\pi n \pm \pi/4$ where $n = 0, 1, 2, 3 \ldots$

now

for $n = 0$ for positive

$$\theta - 36.9^\circ = \pi/4$$

$$\theta = \pi/4 + 36.9^\circ$$

$$\theta = 45^\circ + 36.9^\circ = 81.9^\circ$$

For $n = 1$

$$\theta - 36.9^\circ = 2\pi + \pi/4$$

$$\theta - 36.9^\circ =$$

$$\theta = 362.1^\circ$$

for negative.

$$\theta = 2\pi n - \pi/4$$

for $n = 1$.

$$\theta - 36.9^\circ = 2\pi - \pi/4$$

$$\theta = 351.9^\circ$$

$\therefore$ the value of of $4\cos\alpha + 3\sin\alpha = 5\sqrt{2}/2$ is 351.9°

Extract 15.1: A sample of correct responses to question 5

In Extract 15.1, the candidate had sufficient knowledge of the topic on trigonometry. He approached each part, following all the necessary steps as well as applying the appropriate concepts accordingly.

On the other side, the analysis of the candidates' responses shows that some candidates had inadequate knowledge and skills on the concepts tested. For instance, in part (a), they applied inappropriate concepts. For example, some candidates used wrong trigonometric identities in trying to prove the given identity. They wrote $\frac{\sin 3\alpha}{\sin\alpha} - \frac{\cos 3\alpha}{\cos\alpha} = \frac{\sin 3\alpha\cos\alpha + \cos 3\alpha\sin\alpha}{\sin\alpha\cos\alpha} = \sin 2\alpha + \cos 2\alpha = 2(\sin^2\alpha + \cos^2\alpha)$. Other candidates confused trigonometric identities with trigonometric equations, as a result they solved the given identities.

In part (b), there were also some candidates who applied wrong identities. They considered the right hand side of the given equation as $\tan 2x$. After defining $\tan 2x$ in terms of $\sin 2x$ and $\cos 2x$, they wrote $\sin 2x = 2\sin 4x\cos x$ and $\cos 2x = 2\cos 4x\cos x$. Others applied wrong mathematical computation; they wrote $\frac{\sin x + \sin 2x + \sin 3x}{\cos x + \cos 2x + \cos 3x}$ as $\frac{\frac{\sin x}{\cos x} + \frac{\sin 2x}{\cos 2x} + \frac{\sin 3x}{\cos 3x}}{1+1+1}$. In part (c), most of the candidates who failed to get the correct answer recalled correctly the concept of inverse of trigonometry but failed to simplify the obtained equation involving β. For example, one candidate obtained $2\beta^2 - \beta = 0$ which led them to obtain $\beta = \frac{1}{2}$ instead of $\beta = \pm\sqrt{\frac{1}{2}}$.

In part (d), some candidates did not use the expression $R\cos(\theta - \alpha)$ in solving the equation $4\cos\theta + 3\sin\theta = \frac{5\sqrt{2}}{2}$ as they were instructed. Some used the identity $\cos^2\theta + \sin^2\theta$ to formulate the quadratic equation in $\sin\theta$ as $4\sqrt{1-\sin^2\theta} + 3\sin\theta = \frac{5\sqrt{2}}{2}$. Those candidates failed to simplify their equation to the correct form. As a result, they obtained $16(1-\sin^2\theta) + 9\sin^2\theta = \frac{5\sqrt{2}}{2}$. Others used t-formula to solve the equation.

Extract 15.2 provides an incorrect solution from one of the candidates who approached this question with wrong interpretation.

5(b) Required to prove that

$$\frac{\sin x + \sin 2x + \sin 3x}{\cos x + \cos 2x + \cos 3x} = \tan 2x$$

Divide through out by cos with respect to each variable

$$= \frac{\frac{\sin x}{\cos x} + \frac{\sin 2x}{\cos 2x} + \frac{\sin 3x}{\cos 3x}}{\frac{\cos x}{\cos x} + \frac{\cos 2x}{\cos 2x} + \frac{\cos 3x}{\cos 3x}}$$

$$= \frac{\tan x + \tan 2x + \tan 3x}{3}$$

$$= \frac{\cancel{3}\tan 6x}{3}$$

$$= \tan 2x$$

Since L.H.S = R.H.S

Hence Proved.

Extract 15.2: A sample of incorrect responses to question 5

In Extract 15.2, the candidate failed to use the factor formula to prove the given identity. Instead, the candidate divided the given equation by *cosine* throughout.

2.2.6 Question 6: Algebra

This question had four parts, (a), (b), (c) and (d). In part (a), the candidates were required to find the numerical values of p and q in the expansion of $\dfrac{1+px+qx^2}{(1-x)^2}$ if the coefficients of x and x^2 are zero. In part (b), the candidates were required to use mathematical induction to prove that for every positive integer, $3^{2n-2}+2^{6n}$ is divisible by 5. In part (c), the candidates were required to use the synthetic method to find the quotient and remainder when $P(x)$ is divided by $x+3$ given that $P(x)=2x^3+7x^2-5$. In part (d), they were required to find the determinant and inverse of the matrix $A=\begin{pmatrix} 2 & 1 & 0 \\ 1 & 5 & 2 \\ 1 & -1 & 1 \end{pmatrix}$ and hence to solve the simultaneous equations

$$\begin{cases} 2x+y=4 \\ x+5y+2z=7 \\ x-y+z=1 \end{cases}.$$

The data analysis shows that 7,525(56.0%) candidates answered this question, out of which 1,361(18.1%) candidates scored 0 to 6.5 marks. While 2,433(32.3%) candidates scored 7 to 11.5 marks and 3,731(49.6%) candidates scored 12 to 20 marks. Generally, the candidates' performance in this question was good as 81.9% of the candidates who attempted the question scored from 7 to 20 marks. Figure 17 illustrates the candidates' performance in this question.

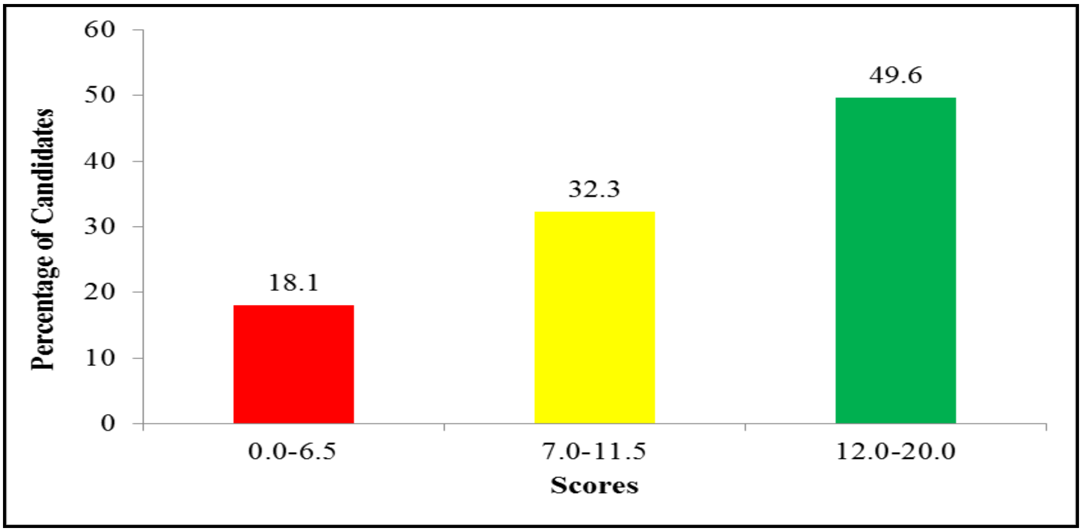


Figure 17: *Candidates' Performance in Question 6*

The candidates who performed well in this question understood the question and had appropriate knowledge on the topic of algebra. In part (a), the candidates transformed the expression $\frac{1+px+qx^2}{(1-x)^2}$ into $(1+px+qx^2)(1-x)^{-2}$. Thereafter, they applied the Binomial theorem to expand $(1-x)^{-2}$ as far as the term in x^2 and obtained $(1-x)^{-2}=1+2x+3x^2+4x^3+\ldots$. Then, they multiplied this expression by $(1+px+qx^2)$ to obtain $1+\left(2+p\right)x+\left(3+2p+q\right)x^2+(4+3p+2q)x^3+\ldots$. Then, they equated $2+p$ equals to zero and $3+2p+q$ equals to zero and obtained the required values of p and q as -2 and 1 respectively. In part (b), the candidates went through all the three required steps of mathematical induction in proving the statement that $3^{2n-2}+2^{6n}$ is divisible by 5. In step (i), they proved that the statement is true by substituting $n=1$ into $3^{2n-2}+2^{6n}$ to obtain $65=5\times13$. In step (ii), they assumed the statement was true for any positive integer k by substituting $n=k$ into $3^{2n-2}+2^{6n}$ to obtain $3^{2k-2}+2^{6k}=5m$ where $m\in N$. In step (iii), they substituted $n=k+1$ into $3^{2n-2}+2^{6n}$ to obtain $3^{2(k+1)-2}+2^{6(k+1)}$. Then, they used step (ii) to show that this statement was also true. They did this by substitution of statement of step (ii) into statement of step (iii). For example, some candidates made 2^{6k} the subject from step (ii) to obtain $2^{6k}=5m-3^{2k-2}$ and substituted this into $3^{2(k+1)}+2^{6(k+1)}$ to obtain $S(k+1)=3^{2(k+1)}+2^{6(k+1)}=5(64m-11.3^{2k-2})$.

In part (c), the candidates were able to use the synthetic division method to find the quotient and remainder. They equated $x+3$ equals to zero and obtained $x=-3$. Then, they extracted the coefficients (2, 7, 0, -5) from the given polynomial function $P(x)=2x^3+7x^2-5$. Finally, they arranged the coefficients in the following form:

$$
\begin{array}{c|ccc|c}
-3 & 2 & 7 & 0 & -5 \\
 & & -6 & -3 & 9 \\
\hline
 & 2 & 1 & -3 & 4
\end{array}
$$

From that arrangement, they obtained the quotient and remainder as $2x^2+x-3$ and 4 respectively.

In part (d), the candidates computed the determinant of the given matrix A correctly by taking the sum of the products of the elements and the corresponding cofactors along the first row as $|A| = 2\begin{vmatrix} 5 & 2 \\ -1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 5 \\ 1 & -1 \end{vmatrix} = 15$. They computed the inverse of matrix A by following all the necessary steps. That is, computing determinant, cofactors, adjoint and then inverse. They computed the cofactors of the matrix and obtained $\begin{pmatrix} 7 & 1 & -6 \\ -1 & 2 & 3 \\ 2 & -4 & 9 \end{pmatrix}$. They computed the adjoint, $adj(A)$ by transposing the matrix of cofactors and then the inverse $A^{-1} = \frac{1}{|A|}adj(A) = \frac{1}{15}\begin{pmatrix} 7 & -1 & 2 \\ 1 & 2 & -4 \\ -6 & 3 & 9 \end{pmatrix}$. Thereafter, they solved the given simultaneous equation using the inverse obtained and ended up with the correct answers such as $x = \frac{23}{15}$, $y = \frac{14}{15}$ and $z = \frac{6}{15}$. Extract 16.1 shows a sample of responses from one of the candidates who performed well.

6. a) $\frac{1+px+qx^2}{(1-x)^2} = (1+px+qx^2)(1-x)^{-2}$.

$= (1+px+qx^2)(1-x)^{-2}$.

on expanding $(1-x)^{-2}$ binomial

$(1+px+qx^2)(1+2x+\frac{-2(-3)}{2!}x^2+ \ldots)$

neglecting higher powers from x^3.

$(1+px+qx^2)(1+2x+\frac{6x^2}{2})$

$(1+px+qx^2)(1+2x+3x^2)$

$1+2x+3x^2+px+2px^2+3px^3+qx^2+2qx^3+3qx^4$

$1+(2+p)x+(3+2p+q)x^2+3px^3+2qx^3+3qx^4$

coefficients of

$x = 2+p$.

$x^2 = 3+2p+q$.

$2+p=0$ ——(1)

$3+2p+q=0$ ——(ii)

Consider eqn (i)

$2+p=0$

$p=-2$

Substitute p to eqn(ii)

$3+2(-2)+q=0$

$3+-4+q=0$

$-1+q=0$

$q=1$.

$\therefore$ $p=-2$ and $q=1$.

b) $3^{2n-2}+2^{6n} = 5m$ (required to prove)

Test if it's true for $n=1$.

$3^{2(1)-2}+2^{6} = 65 = 13\times 5$.

hence it's true for $n=1$

6 b) For $n=2$.

$3^{2(2)-2}+2^{6(2)} = 4105 = 821\times5$

hence it's true. for $n=2$

Assume it's true for $n=k$.

$3^{2k-2}+2^{6k} = 5m$

Then for $n=k+1$.

$3^{2(k+1)-2}+2^{6(k+1)} = 5m$

Consider LHS.

$3^{2k+2-2}+2^{6k+6} = 3^{2k}+2^{6k+6}$

$3^{2k}+2^{6k+6}$ —— (i)

from $n=k$.

$2^{6k} = 5m - 3^{2k-2}$

Substitute 2^{6k} to eqn (i).

$3^{2k}+2^{6k}.2^{6}$ -

$3^{2k}+(5m-3^{2k-2})2^{6}$.

$3^{2k}+5m.2^{6}-2^{6}(3^{2k}.3^{-2})$.

$3^{2k}-3^{2k}(2^{6}.3^{-2})+5m.2^{6}$

$3^{2k}(1-(2^{6}.3^{-2}))+2^{6}\times5m$

$3^{2k}\left(\frac{-55}{9}\right)+2^{6}m\times5$.

$\left(3^{2k}\left(-\frac{11}{9}\right)+2^{6}m\right)5. = 5m.$

where m is any number.

$\therefore\ 3^{2(k+1)-2}+2^{6(k+1)} = 5\left(3^{2k}\left(-\frac{11}{9}\right)+2^{6}m\right)$

but $k+1=n$.

$3^{2n-2}+2^{6n} = 5m$.

hence proved!

6 c) $P(x) = 2x^3 + 7x^2 - 5$.

$x + 3 = 0$.

$x = -3$.

$$\begin{array}{r|rrrr} -3 & & & & \\ & 2 & 7 & 0 & -5 \\ & 0 & -6 & -3 & 9 \\ \hline & 2 & 1 & -3 & 4 \end{array}$$

$\underbrace{2 \quad 1 \quad -3}_{\text{Quotient}}$ | 4 → remainder.

∴ Remainder is 4. and Quotient of the equation is $2x^2 + x - 3$

d) $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 5 & 2 \\ 1 & -1 & 1 \end{pmatrix}$

determintait. $|A|$.

$|A| = 2|5+2| - 1|1-2| + 0|-1-5|$

$= 2(7) + 1. + 0$

$|A| = 15$.

∴ Determinant of A is 15

~~Fro~~:

Co-factor of A

$$A^c = \begin{pmatrix} \begin{vmatrix} 5 & 2 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} \end{pmatrix}$$

6 d) $A^C = \begin{pmatrix} 7 & 1 & -6 \\ -1 & 2 & 3 \\ 2 & -4 & 9 \end{pmatrix}$

$A^T = \begin{pmatrix} 7 & -1 & 2 \\ 1 & 2 & -4 \\ -6 & 3 & 9 \end{pmatrix}$.

$A^{-1} = \frac{1}{|A|} \text{Adj}(A)$.

$\therefore A^{-1} = \frac{1}{15} \begin{pmatrix} 7 & -1 & 2 \\ 1 & 2 & -4 \\ -6 & 3 & 9 \end{pmatrix}$

~~From~~;

$\therefore$ Inverse of $A = \begin{pmatrix} 7/15 & -1/15 & 2/15 \\ 1/15 & 2/15 & -4/15 \\ -2/5 & 1/5 & 3/5 \end{pmatrix}$.

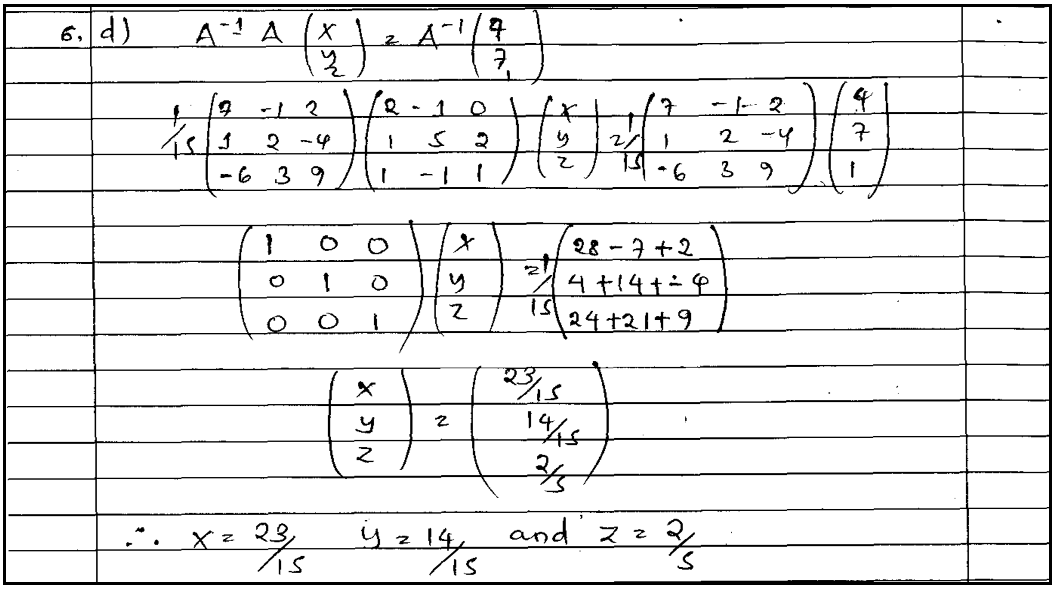
From;

$$\begin{cases} 2x + y = 4 \\ x + 5y + 2z = 7 \\ x - y + z = 1 \end{cases}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 5 & 2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 1 \end{pmatrix}.$$

but.

Inverse of matrix $= \frac{1}{15} \begin{pmatrix} 7 & -1 & 2 \\ 1 & 2 & -4 \\ -6 & 3 & 9 \end{pmatrix} = A^{-1}$.



6. d) $A^{-1}A\begin{pmatrix}x\\y\\z\end{pmatrix} = A^{-1}\begin{pmatrix}4\\7\end{pmatrix}$

$\frac{1}{15}\begin{pmatrix}7&-1&2\\1&2&-4\\-6&3&9\end{pmatrix}\begin{pmatrix}2&-1&0\\1&5&2\\1&-1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{15}\begin{pmatrix}7&-1&2\\1&2&-4\\-6&3&9\end{pmatrix}\begin{pmatrix}4\\7\\1\end{pmatrix}$

$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{15}\begin{pmatrix}28-7+2\\4+14+-4\\24+21+9\end{pmatrix}$

$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}23/15\\14/15\\2/5\end{pmatrix}$

$\therefore\ x = 23/15,\quad y = 14/15$ and $z = 2/5$

Extract 16.1: A sample of correct responses to question 6

In Extract 16.1, the candidate used the binomial theorem to calculate the numerical values of p and q. He/she also used mathematical induction to prove that the given statement is divisible by 5. Likewise, the candidate was able to find the remainder of the function by synthetic division. Finally, the candidate found the determinant and inverse of the given matrix which used to solve the simultaneous equation by inverse method.

On the other hand, the analysis reveals that, there were candidates who did not understand how to solve this question. In part (a), the candidates failed to transform $\frac{1+px+qx^2}{(1-x)^2}$ into $(1+px+qx^2)(1-x)^2$ instead they partilised the given expression by letting $u=1-x$ to obtain $\frac{1}{u^2}+\frac{px}{u^2}+\frac{qx^2}{u^2}=\frac{1}{(1-x)^2}+\frac{px}{(1-x)^2}+\frac{qx^2}{(1-x)^2}$. Then, they equated $\frac{p}{(1-x)^2}$ and $\frac{q}{(1-x)^2}$ to zero and obtained wrong values of $p=0$ and $q=0$. Other candidates did not apply the Binomial theorem to expand the given expression $\frac{1+px+qx^2}{(1-x)^2}$. They instead equated $1-px+qx^2$ to $(1-x)^2$ and set $x=1$ and $x=-1$ in trying to solve for p and q, which resulted in wrong values of p and q as -1 and 0 respectively. Furthermore, there were candidates who equated the expression

$\frac{1+px+qx^2}{(1-x)^2}$ to 1, aiming to solve for p and q. This could not help them as they failed even to solve the equation they had formulated.

In part (b), the candidates failed to write the general assumption in step (ii). They wrote $3^{2k-2}+2^{6k}=5$ instead of $3^{2k-2}+2^{6k}=5m$. They also wrote a wrong statement in step (iii) as $3^{2(k+1)-2}+2^{6(k+1)}=5$ instead of $3^{2(k+1)-2}+2^{6(k+1)}$. Using these statements and improper substitutions ended up with wrong result $2^{6k+6}+3^{-1}=1$.

In part (c), some candidates did not extract the coefficients from the given polynomial equation, they wrote:

$$\begin{array}{rrrr|l} 2x^3+7x^2+0x-5 & & & & -3 \\ + & -6 & -39 & -117 & \\ \hline 2 & 13 & 39 & 112 & \end{array}$$

They also used subtraction instead of addition. This made the candidates get the wrong results $13x^2+39x+112$ and 0 as quotient and remainder respectively. In part (d), the candidates did not understand the question. Some of them solved the question using inappropriate techniques such as substitution method instead of inverse method while others used Crammer's rule in their solution. Other candidates applied wrong concepts in finding the inverse matrix. They calculated the inverse A^{-1} by multiplying the determinant by the coefficient matrix, that is, $A^{-1}=\frac{1}{15}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 5 & 2 \\ 1 & -1 & 1 \end{pmatrix}$. Also, there were other candidates who used the concepts of determinant in finding the cofactors of matrix A. They wrote:

$$A_c=\begin{pmatrix} 2\begin{vmatrix} 5 & 2 \\ -1 & 1 \end{vmatrix} & -1\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} & 0\begin{vmatrix} 1 & 5 \\ 1 & -1 \end{vmatrix} \\ -1\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} & 5\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} & -2\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \\ 1\begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} & 1\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} & 2\begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} \end{pmatrix}$$

and lastly they failed to determine the correct answer.

Extract 16.2 illustrates an incorrect response from one of the candidate who attempted the question in part (a).

Solution

06 (a) $\frac{1 + px + qx^2}{(1-x)^2}$

If the coefficients of x and x^2 are zero then

$$\frac{1 + px + qx^2}{(1-x)^2} = 1.$$

$$1 + px + qx^2 = (1-x)^2.$$

$$1 + px + qx^2 = (1-x)(1-x)$$
$$1 + px + qx^2 = 1 - x - x + x^2.$$

$$1 + px + qx^2 = 1 - 2x + x^2.$$

Upon comparing.

$$1 = 1.$$
$$p\cancel{x} = -2\cancel{x}$$
$$\therefore p = -2$$

$$q\cancel{x^2} = \cancel{x^2}$$
$$q = 1.$$

$\therefore$ The numerical values of p and $q = 1$.

$$\therefore p = -2$$
$$q = 1.$$

Extract 16.2: A sample of incorrect responses to question 6(a)

In Extract 16.2, the candidate failed to use the Binomial theorem to calculate the numerical value of p and q. Thus, he/she used inappropriate approach and failed to get the correct answer.

2.2.7 Question 7: Differential Equations

The question comprised four parts, (a), (b), (c) and (d). Part (a) (i), required the candidates to determine the most general function $M(x, y)$ such that the differential equation $M(x, y)dx + (2x^2y^3 + x^4y)dy = 0$ is exact. And (a) (ii), required the candidates to solve the differential equation $(xy + x)dx - (x^2y^2 + x^2 + y^2 + 1)dy = 0$ by separating the variables. In part (b), the candidates were asked to find the general solution of the differential equation $\cos x \frac{d^2y}{dx^2} - \sin x \frac{dy}{dx} = 0$. In part (c), the question stated that: A liquid of $72^\circ C$ placed in a room at $25^\circ C$ has a temperature of $65^\circ C$ after 5 minutes. Then, the candidates were asked to find its temperature after further 10 minutes. Part (d), required the candidates to formulate a differential equation of a circle which passes through the origin and whose centre lies on the y-axis.

The question was attempted by 1,649 candidates (12.3%), whereby 35.4 per cent scored from 0 to 6.5 marks, 38.8 per cent scored from 7 to 11.5 marks and 25.8 per cent scored from 12 to 20 marks. The candidates' performance in this question was good, as 64.6 per cent of the candidates scored from 7 to 20 marks. Figure 18 provides a summary of the candidates' performance in this question.

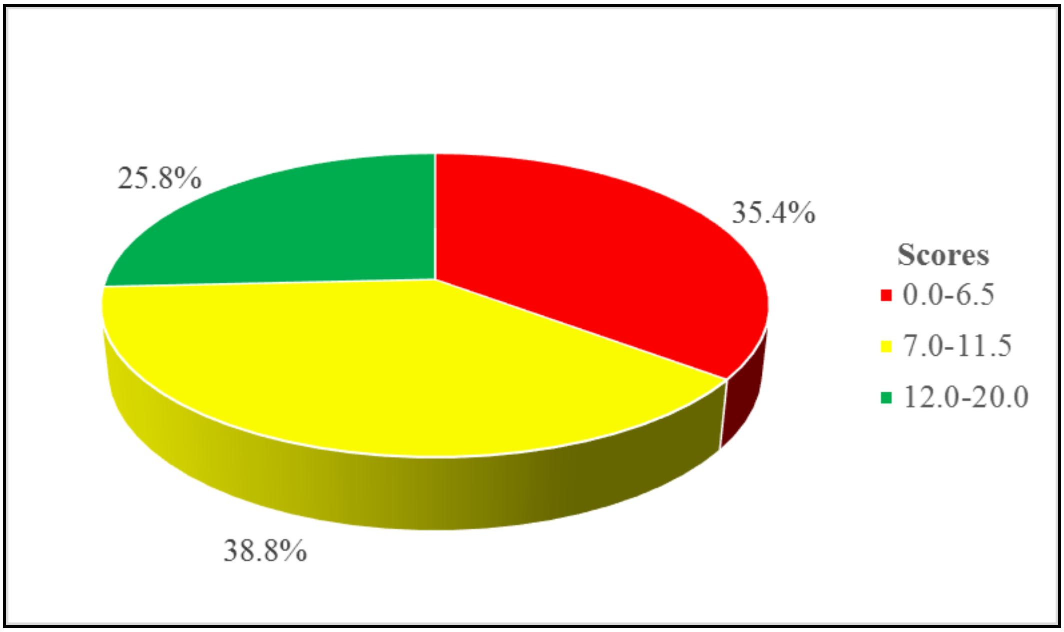


Figure 18: *Candidates' Performance in Question 7*

The analysis of data shows that, among 1,649 candidates who attempted this question, there were 23 (1.4%) candidates who scored 20.0 marks while 30 (1.8%) candidates scored 0 mark.

The candidates who attempted the question correctly had adequate knowledge of differential equations. In part (a) (i), they were able to state the conditions for differential equations to be exact, that is, $\frac{\partial N}{dx}(x,y) = \frac{\partial M}{dy}(x,y)$ thus $\frac{\partial N}{dx}(x,y) = 4xy^3 + 4x^3y$. Then, they managed to apply integration skills on $\partial M(x,y) = \left(4xy^3 + 4x^3y\right)dy$ to get the general function $M(x,y) = xy^4 + 2x^3y^2 + A$ as per the question's instructions. In part (a) (ii), the candidates managed to separate variables by collecting like terms $\left(\frac{x}{x^2+1}\right)dx = \left(\frac{y^2+1}{y+1}\right)dy$ and applying the concept of integration to get the required equation, $\frac{1}{2}\ln\left(x^2+1\right) = \frac{y^2}{2} - y + 2\ln\left(y+1\right) + c$. Also, in part (b), candidates were able to reduce the second order differential equation into first order differential equation as $\cos x\frac{d^2y}{dx^2} - \sin x\frac{dy}{dx} = 0 \Rightarrow \cos x\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{d}{dx}\left(\cos x\right)\frac{dy}{dx} = 0$. Then, they evaluated $\frac{d}{dx}\left(\cos x\frac{dy}{dx}\right) = 0$ to get $y = c\ln(\sec x + \tan x) + A$ which was the required solution.

In part (c), the candidates were able to apply skills of differential equation as well as integration skills by using the given data $t = 0$ minute, $\theta = \theta_0 = 72^0C$ into equation $\frac{d\theta}{dt} = -k\left(\theta - 25\right)$ to obtain $k = -\frac{1}{5}\ln\left(\frac{40}{47}\right)$. Then, they computed temperature after 10 minutes to get 53.97^0C, as required .They finally recalled the general equation of a circle in part (d), which was $x^2 + y^2 + 2gx + 2fy + c = 0$. Then, they reduced the equation to $x^2 + y^2 + 2fy = 0$ since it lies on $y-$axis. Thereafter, they applied the differentiation skills to get $f = \frac{-\left(x + y\frac{dy}{dx}\right)}{\frac{dy}{dx}}$ and substituted f into reduced equation of circle to get the required equation $\left(x^2-y^2\right)\frac{dy}{dx}-2xy=0$. Extract 17.1 shows the response of a candidate who had adequate skills on the concept tested.

7. (a) (i) $M(x,y)\,dx + (2x^2y^3 + x^4y)\,dy = 0$

Now;

$M\,dx + (2x^2y^3 + x^4y)\,dy = 0$

So If exact,

$$\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$$

$$\frac{\partial M}{\partial y} = \frac{\partial(2x^2y^3 + x^4y)}{\partial x}$$

$$\frac{\partial M}{\partial y} = 2y^3(2x) + y(4x^3)$$

$$\frac{\partial M}{\partial y} = 4xy^3 + 4x^3y$$

$$\int \partial M = \int (4xy^3 + 4x^3y)\,\partial y$$

7 (a)(i) So,

$$M = \int 4xy^3\,\partial y + \int 4x^3y\,\partial y$$

$$M = \frac{4xy^4}{4} + \frac{4x^3y^2}{2} + c$$

$$M = xy^4 + 2x^3y^2 + C$$

So hence;

$$M(x,y) = xy^4 + 2x^3y^2 + C.$$

7 (b) $\cos x \frac{d^2y}{dx^2} - \sin x \frac{dy}{dx} = 0$

$$\cos x \frac{d^2y}{dx^2} + -\sin x \frac{dy}{dx} = 0$$

$$\frac{d\left(\cos x \frac{dy}{dx}\right)}{dx} = 0$$

$$\int d\left(\cos x \frac{dy}{dx}\right) = \int 0\, dx$$

$$\cos x \frac{dy}{dx} = A$$

$$dy = \frac{A}{\cos x}\, dx$$

$$\int dy = \int A \sec x\, dx$$

$$y = A \int \sec x\, dx$$

$$y = A \ln(\sec x + \tan x) + C$$

$\therefore$ The general solution is

$$y = A \ln(\sec x + \tan x) + C.$$

Extract 17.1: A sample of correct responses to question 7(a) (i) and (b)

In Extract 17.1, part (a) (i), the candidate determined the general function $M(x, y)$ correctly. Furthermore, in part (b), the candidate was able to solve the second order differential equation and got the general solution.

Even though most of the candidates attempted this question correctly, there were others who had no enough skills and knowledge of differentiation and integration to solve the differential equations. In part (a) (i), the candidates were not able to use partial derivatives correctly to justify whether differential equation was exact or not. For instance, some wrote: $\frac{dN}{dx} = 4xy^3 + 3x^3y$ and $\frac{dM}{dy} = 4xy^3 + 3x^3y$ from which they got the wrong functions such as $M(x, y) = xy^4 + \frac{3}{2}x^3y^2$ and $M(x, y) = 2x^2y^2$ instead of

$M(x,y)=xy^4+2x^3y^2+c$. In part (a) (ii), they failed to separate the variables. For example, a few of them wrote $\int\frac{dy}{(y^2+1)(y+1)}=\int\frac{xdx}{x^2+1}$ instead of $\int\left(\frac{y^2+1}{y+1}\right)dy=\int\left(\frac{x}{x^2+1}\right)dx$. Those candidates ended up with incorrect equation of $\tan^{-1}y-\frac{1}{2}\ln\left(1+y^2\right)+\ln\left(1+y\right)=\ln\left(x^2+1\right)+c$. Not only that but also some used the substitution $y=ux$. Then, they arranged it in the form $\frac{dy}{dx}=u+x\frac{du}{dx}$. From this step, they substituted the values obtained into $\left(xy+x\right)dx-\left(x^2y^2+x^2+y^2+1\right)dy=0$ and concluded the solution to this differential equation as $\frac{du}{dx}=\frac{x-x^4u^3-u^3-u}{\left(x^4u^2+x^2+u^2x^2+1\right)x}$ which was wrong and not even required. In addition to that, some candidates used the concept of integrating factor $R(x)=e^{sp(y)dy}$ instead of separating variables which made them get incorrect answers such as $\frac{y^2}{2}+y=\frac{1}{2}\ln(x^2+1)+c$ and $y=-\ln(\cos x)+c$.

In part (b), a few candidates failed to reduce second order differential equation into its first order. They substituted m into $\cos x\frac{d^2y}{dx^2}-\sin x\frac{dy}{dx}=0$ to get $m^2\cos x-m\sin x=0$. Then, they solved the equation by using the general quadratic formula, which resulted to incorrect solution $y=\left(Ax+B\right)e^{x\tan x}$. Also, others used t-substitutions using $\cos x=\frac{1-t^2}{1+t^2}$ and $\sin x=\frac{2t}{1+t^2}$ into $m^2\cos x-m\sin x=0$ and got $y=A\left(\ln\left(1-\tan^2\left(\frac{x}{2}\right)\right)\right)+c$. In part (c), the candidates failed to apply the Newton's law of cooling to obtain the temperature of the liquid after 10 minutes. For example, they used the formula $T_t=T_s-(T_0-T_s)e^{-k}$. Then, they substituted $T_0=72c, T_s=25c, T_t=65c, t_1=5c, t_2=10c$ to obtain $65=25-7e^{-5k}$ which led to $k=\ln\left(\frac{8}{7}\right)$. By using the value of k, they obtained the temperature after 10 minutes such as $23.158^\circ C$, $50^\circ C$, $35.99^\circ C$ instead of $53.97^\circ C$. In part (d), some candidates used the equation $x^2+y^2=r^2$ that led to incorrect differential equation of

$y\frac{dy}{dx}+x=0$. Extract 17.2 shows an incorrect response from one of the candidates who attempted this question.

7 (a) i) solution

Given

$M(x,y)\,dx + (2x^2y^2 + x^4y)\,dy = 0$

since it is exact

$\cancel{M_x}$ = let $2x^2y^2 + x^4y$ be N

$\frac{\partial N}{\partial x} = 4xy^2 + 4x^3y.$

$N(\cancel{x},y) = \int(2x^2y^2 + x^4y)\,dy$

$M(x,y) = \frac{2x^2y^3}{3} + \frac{x^4y^2}{2}$

let $M(x,y) = \frac{2x^2y^3}{3}$

$\frac{\partial M}{\partial y} = \frac{2x^2}{3}\cdot 3y^2 = 2x^2y^2.$

let $M(x,y) = \frac{x^4y^2}{2}$

$\frac{\partial M}{\partial y} = \frac{x^4}{2}\cdot 2y = xy^4$

also $M = \int 2x^2y^2\,dx = \frac{2x^3y^2}{3}$

$\therefore M(x,y) = 2x^2y^2$

$\therefore$ The value of the function

$M(x,y) = 2x^2y^2.$

7. a) ii) $(xy+x)dx - (x^2y^2+x^2+y^2+1)dy = 0$

$$(x^2y^2+x^2+y^2+1)\frac{dy}{dx} = (xy+x)\frac{dx}{dx}$$

$$(x^2y^2+x^2+y^2+1)\frac{dy}{dx} = xy+x$$

let. $y = ux$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$\left(x^2(ux)^2+x^2+(ux)^2+1\right)\left(u+x\frac{du}{dx}\right) = x^2u+x$$

$$(x^4u^2+x^2+u^2x^2+1)\left(u+x\frac{du}{dx}\right) = x^2u+x$$

$$u+x\frac{du}{dx} = \frac{x^2u+x}{x^4u^2+x^2+u^2x^2+1}$$

$$x\frac{du}{dx} = \frac{x^2u+x}{x^4u^2+x^2+u^2x^2+1} - u$$

$$x\frac{du}{dx} = \frac{x^2u+x-(x^4u^3+ux^2+u^3+u)}{x^4u^2+x^2+u^2x^2+1}$$

$$x\frac{du}{dx} = \frac{x^2u+x-x^4u^3-ux^2-u^3-u}{x^4u^2+x^2+u^2x^2+1}$$

$$x\frac{du}{dx} = \frac{-x^4u^3-u^3-u+x}{x^4u^2+x^2+u^2x^2+1}$$

$$\frac{du}{dx} = \frac{x-x^4u^3-u^3-u}{(x^4u^2+x^2+u^2x^2+1)x}$$

7. b) $\cos x \frac{d^2y}{dx^2} - \sin x \frac{dy}{dx} = 0$

$$\frac{d^2y}{dx^2} - \tan x \frac{dy}{dx} = 0$$

let. $p = \frac{dy}{dx}$

$$\frac{d^2y}{dx^2} = \frac{dp}{dx}$$

$$\frac{dp}{dx} - p\tan x = 0$$

$$dp = p\tan x\, dx$$

$$\int \frac{1}{p}\, dp = \int \tan x\, dx$$

$$\cancel{\ln} p = \cancel{\ln} \cos x$$

$$p = \cos x.$$

$$\frac{dy}{dx} = \cos x$$

$$\int dy = \int \cos x\, dx$$

$$y = \sin x + c.$$

$$\therefore\ y = \sin x + c.$$

Extract 17.2: A sample of incorrect responses to question 7

In Extract 17.2, in part (a), the candidate was not able to determine the correct general function $M(x, y)$ and also failed to solve the given differential equation by separating the variables. Also, in part (b), he/she was not able to solve the second order differential equation.

2.2.8 Question 8: Coordinate Geometry II

The question stated:

(a) Find the coordinates of foci, the vertices, the eccentricity and the length of the latus rectum of the hyperbola $16x^2 - 9y^2 = 576$.

(b) (i) Determine the equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $P(a\cos\theta, b\sin\theta)$.

(ii) If the normal in part (b) (i) meets the x-axis at A and the y-axis at B, find the area of the triangle AOB where O is the origin.

(c) Show that the equation of the tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is $x - ty + at^2 = 0$

(d) (i) Change the Cartesian equation $(x^2 + y^2)^3 - 2xy(x^2 - y^2)$ into a polar equation.

(ii) Sketch the graph of $r = 1 - 2\cos\theta$ from $\theta = 0$ to $\theta = 2\pi$.

The data analysis shows that, 7,287 (54.2%) candidates opted this question whereby 526 (7.2%) candidates scored from 0 to 6.5 marks. Moreover, the data shows that 2,201 (30.2%) candidates scored from 7 to 11.5 marks and 4,560 (62.6%) candidates scored from 12 to 20 marks. Generally, the candidates' performance in this question was good as shown in Figure 19.

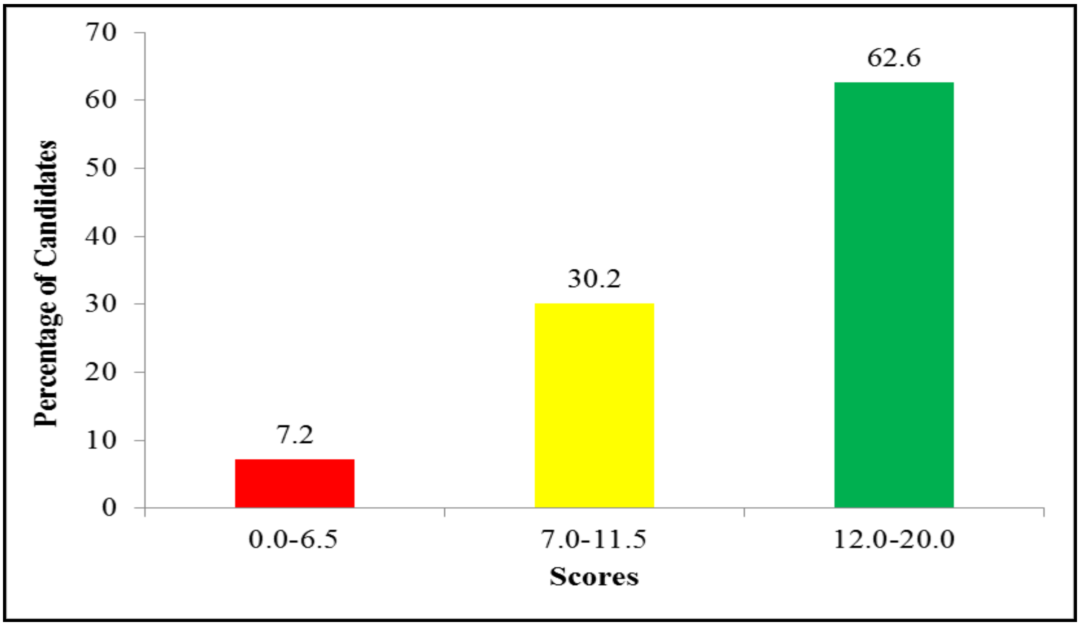


Figure 19: *Candidates' Performance in Question 8*

Further analysis depicts that, among the candidates who opted for the question, 2.5 per cent scored 20 marks.

The candidates who performed well were able to divide the equation $16x^2 - 9y^2 = 576$ to get $\frac{x^2}{36} - \frac{y^2}{64} = 1$ which was compared to the general equation of the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in part (a). Then, they obtained the coordinates of foci, vertices, eccentricity and length of latus rectum as $(\pm 10, 0)$, $(\pm 6, 0)$, $\frac{5}{3}$ and 21.33 units respectively as per requirement. In part (b) (i), the candidates were able to compute the slope of tangent to the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $P(a\cos\theta, b\sin\theta)$ as $m_n = \frac{a\sin\theta}{b\cos\theta}$. Then, they used this slope and contact point $P(a\cos\theta, b\sin\theta)$ to obtain normal equation as $y = \frac{a\sin\theta}{b\cos\theta}x + b\sin\theta - \frac{a^2}{b}\sin\theta$. In part (b) (ii), they were able to obtain $\left(\frac{a^2 - b^2}{a}\cos\theta, 0\right)$ and $\left(0, \ \frac{b^2 - a^2}{b}\sin\theta\right)$ as the coordinates of A and B respectively. Finally, the obtained coordinates enhanced them to calculate the area of the triangle AOB as $A = \frac{1}{2}\frac{\left(b^2 - a^2\right)\left(a^2 - b^2\right)}{ab}\sin\theta\cos\theta$.

In part (c), they were able to use chain rule to get the gradient of the curve at the point of contact as $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2a \times \frac{1}{2at} = \frac{1}{t}$. Then using the gradient obtained they found the equation of the tangent as $\frac{y - 2at}{x - at^2} = \frac{1}{t}$ hence they were able to show that $x - ty + at^2 = 0$. In part (d) (i), the candidates defined correctly the parametric value for $x = r\cos\theta$ and $y = r\sin\theta$. Then, they substituted them directly into expression $\left(x^2 + y^2\right)^3 - 2xy\left(x^2 - y^2\right)$ to get $r^6 - r^4\cos 2\theta \sin 2\theta = r^4(r^2 - \cos 2\theta \sin 2\theta)$. In part (d) (ii), they were able to prepare a table of values for $r = 1 - 2\cos\theta$ correctly which was used to sketch the graph from $\theta = 0$ to $\theta = 2\pi$. Extract 18.1 shows a sample of correct responses from one of the candidates.

8. (a) Now for foci, S

From; $S = (\pm ae, 0)$.

$S = (\pm(6 \times \frac{5}{3}), 0)$ since $a = 6$ and $e = \frac{5}{3}$.

$S = (\pm 10, 0)$.

$S = (10, 0)$ and $(-10, 0)$

$\therefore$ <u>The coordinates of foci are $(10,0)$ and $(-10,0)$.</u>

Now for vertices, V

From; $V = (\pm a, 0)$. where $a = 6$.

$V = (\pm 6, 0)$. $= V(6,0)$ and $V(-6,0)$

$\therefore$ <u>The coordinates of vertices are $(6,0)$ and $(-6,0)$.</u>

For length of latus rectum, L_r

from; $L_r = \frac{2b^2}{a}$

where $b^2 = 64$ and $a = 6$.

Now; $L_r = \frac{2 \times 64}{6}$.

$L_r = \frac{64}{3}$ units

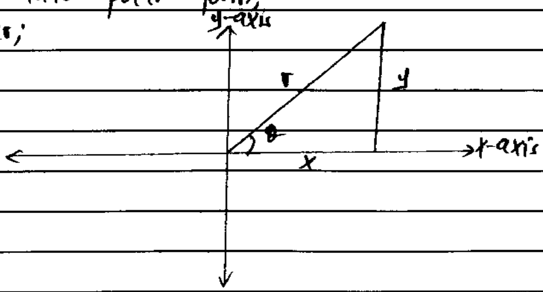
$\therefore$ <u>The length of latus rectum is $\frac{64}{3}$ units.</u>

(d). <u>Solution.</u>

Given; $(x^2 + y^2)^3 - (2xy)(x^2 - y^2)$

Required: Into polar form;

Consider;



From; $\cos\theta = \frac{x}{r}$

$x = r\cos\theta$ --- (i)

Also; $\sin\theta = \frac{y}{r}$

8. (d)(i) $y = r\sin\theta$ --- (ii)

Now from $(x^2+y^2)^3 - 2xy(x^2-y^2)$ --- (iii)

Substitute (i) and (ii) into (iii)

By substituting;

$$(x^2+y^2)^3 - 2xy(x^2-y^2) = \left((r\cos\theta)^2 + (r\sin\theta)^2\right)^3 - 2r\cos\theta\sin\theta\left((r\cos\theta)^2 - (r\sin\theta)^2\right)$$

$$= (r^2\cos^2\theta + r^2\sin^2\theta)^3 - 2r\sin\theta\cos\theta\,(r^2\cos^2\theta - r^2\sin^2\theta).$$

$$= \left(r^2(\cos^2\theta + \sin^2\theta)\right)^3 - 2r\sin\theta\cos\theta\left(r^2(\cos^2\theta - \sin^2\theta)\right).$$

but $\cos^2\theta + \sin^2\theta = 1$ and $\cos^2\theta - \sin^2\theta = \cos 2\theta$

Now;

$$= (r^2)^3 - 2r\sin\theta\cos\theta\,(r^2\cos 2\theta).$$

$$= r^6 - r(2\sin\theta\cos\theta)(r^2\cos 2\theta)$$

but $2\sin\theta\cos\theta = \sin 2\theta$

$$= r^6 - (r\sin 2\theta)(r^2\cos 2\theta)$$

$$= r^6 - r^3\sin 2\theta\cos 2\theta$$

$$= r^3(r^3 - \sin 2\theta\cos 2\theta).$$

$\therefore$ $(x^2+y^2)^3 - 2xy(x^2-y^2)$ into polar equation is $r^3(r^3 - \sin 2\theta\cos 2\theta)$.

(ii). Solution.

Given;

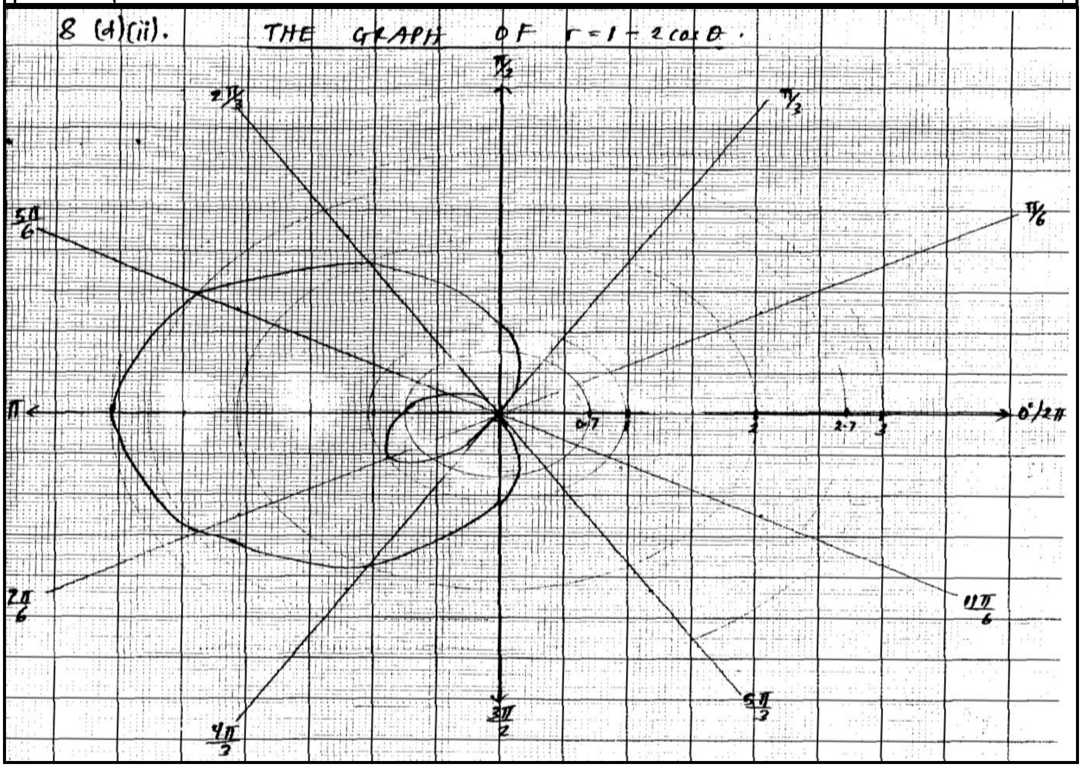
$r = 1 - 2\cos\theta$

Range: $0 \le \theta \le 2\pi$ = $0 \le \theta \le 260^\circ$.

Required: Graph of r

Consider table of values; of $r = 1 - 2\cos\theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{2\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	-1	-0.7	0	1	2	2.7	3	2.7	2	1	0	-0.7	-1



Extract 18.1: A sample of correct responses to question 8(a) and (d)

In Extract 18.1, part (a), the candidate was able to find the foci, vertices, eccentricity and length of latus rectum from the given hyperbola. Likewise, in part (d), (i) he/she changed the Cartesian equation into polar equation and lastly in (ii) sketched the correct graph of the given polar equation.

Apart from those candidates who performed well this question, there were some candidates who faced challenges due to lack of knowledge and skills on Coordinate Geometry. In part (a), there were a number of candidates who used the concepts of the hyperbola which lies on $y-$axis instead of $x-$axis. For this case, they got incorrect values of foci, vertices, eccentricity and length of latus rectum such as $(0,\pm 10)$,$(0,\pm 10)$ $(0,\pm 8)$, $\frac{5}{4}$ and $\frac{9}{8}$ units respectively. Furthermore, a few candidates used $\frac{5}{4}$ $c^2 = b^2 - a^2$ and got incorrect answers. For example, they showed $\text{foci} = (\pm\sqrt{28}, 8)$, $\text{vertices} = (\pm 8, 0)$, $\text{eccentricity} = \frac{\sqrt{28}}{8}$ and length of latus rectum equals to 18.

In part (b) (i), the candidates were unable to apply the concept of perpendicular lines that made them get incorrect equation of the normal such as $yxb^2 + a^3 y\cos\theta - a^2 yx - b^3 x\sin\theta = 0$, $b^2 x\sin\theta - a^3 y\cos\theta + a^2 xy - b^2 x = 0$ and $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = c^2$. Others showed that $\frac{ax}{\cos\theta(a^2+b^2)} - \frac{by}{\sin\theta(a^2+b^2)} = c^2$. In part (b) (ii), a few candidates used incorrect coordinates such as $A\left(\frac{c^2}{a}\cos\theta, 0\right)$ and $B\left(0, \frac{c^2}{a}\sin\theta\right)$ to find the area of a triangle and ended up with wrong results like $A = \frac{c^4 \sin\theta\cos\theta}{2ab}$ and $A = \frac{1}{2}\left(\frac{b^3}{a}\cos\theta\sin\theta - \frac{\sin\theta}{a} - a\cos\theta\sin\theta + \frac{a^3\cos\theta\sin\theta}{b^2}\right)$.

In part (c), some candidates used the point $(0, at^2)$ as the focus of the parabola, foot of perpendicular $(-x, 2at)$ and the moving point $(at^2, 2at)$ on the parabola. Then, they compared the distance from the focus and foot of perpendicular to obtain $x - 2at + at^2 = 0$ instead of $x - ty + at^2 = 0$. In part (d) (i), a few candidates substituted the formula $r^2 = x^2 + y^2$ into $(x^2+y^2)^3 - 2xy(x^2 - y^2)$ to obtain the incorrect answer $(x^2+y^2)^3 - 2xy(x^2-y^2) = (r^2)^3 + 2xy^2$ or $r^6 - 2r^2(r^2 + x^2y^2)$. In part (d) (ii), some candidates were able to prepare table of values but they were not able to sketch the graph of $r = 1 - 2\cos\theta$. Extract 18.2 provides a sample of incorrect responses from a certain candidate.

(a)(ii) solution

$$(x^2+y^2)^3 - 2xy(x^2+y^2).$$

from $r = \sqrt{x^2+y^2}$

$$r^2 = x^2+y^2.$$

$$(r^2)^3 - 2xy(r^2)$$

$$r^6 - 2xyr^2.$$

$$x = \sqrt{r^2-y^2}$$

$$r^6 - 2\left[(\sqrt{r^2-y^2})\sqrt{(r^2-x^2)}\right]r^2$$

$$r^6 - 2\left[\sqrt{(r^2-y^2)(r^2-x^2)}\right]r^2$$

$$r^6 - 2r^2\left[\sqrt{r^4 - r^2x^2 - r^2y^2 + x^2y^2}\right]$$

$$r^6 - 2r^2\left[\sqrt{r^4 - r^2(x^2+y^2) + x^2y^2}\right]$$

(d) (i) $r^6 - 2r^2\left[\sqrt{r^4 - r^2 + x^2y^2}\right]$

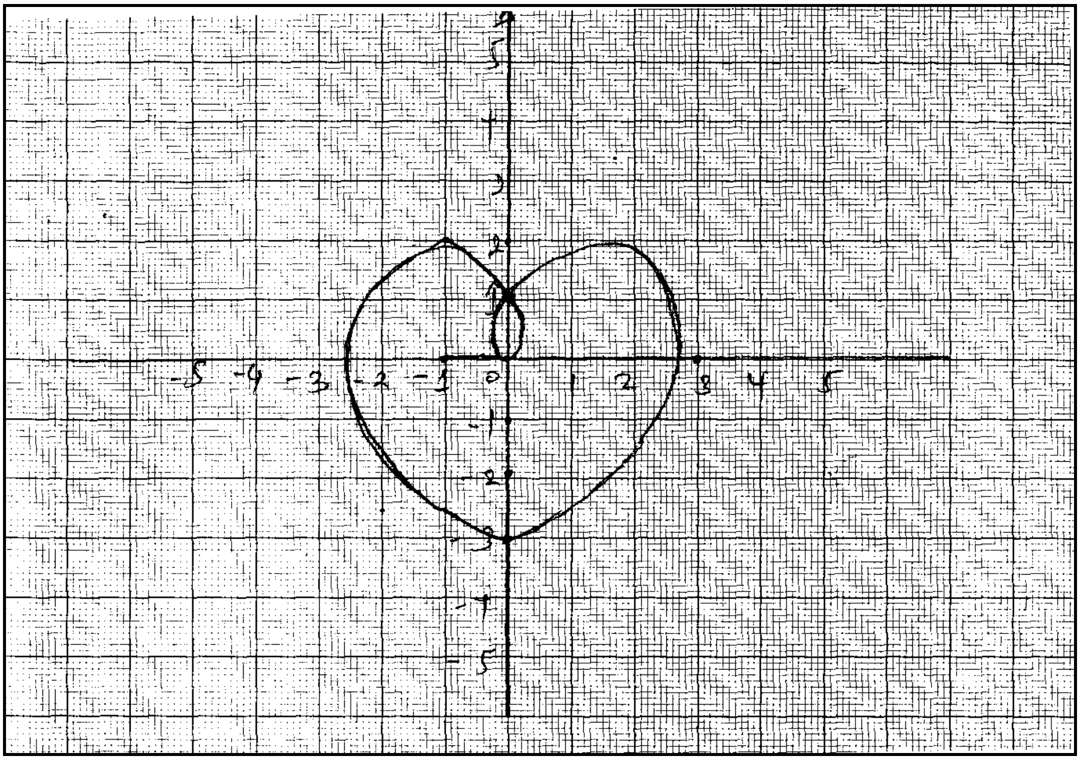
$$r^6 - 2r^2\left[\sqrt{r^2 + x^2y^2}\right].$$

$$r^6 - 2r^2\left[\sqrt{r^2 + (r^2-y^2)(r^2-x^2)}\right]$$

$$r^6 - 2r^2\left[r^2 + r^4 - r^2x^2 - r^2y^2 + x^2y^2\right]$$

$$r^6 - 2r^2\sqrt{r^2 + x^2y^2}$$

$\therefore r^6 - 2r^2\left[\sqrt{r^2 + x^2y^2}\right]$ Hence shown!



Extract 18.2: A sample of incorrect responses to question 8(d)

In Extract 18.2, the candidate was not able to change the Cartesian equation into polar equation and lastly in (ii), he/she failed to sketch the graph of the given polar equation.

3.0 ANALYSIS OF CANDIDATES' PERFORMANCE IN EACH TOPIC

The candidates were examined on 18 topics out of which 10 topics were tested in Advanced Mathematics 1 and eight (8) topics in Advanced Mathematics 2. Advanced Mathematics 1 consisted of ten (10) questions and Advanced Mathematics 2 consisted of eight (8) questions. The topics examined from Advanced Mathematics 1 includes; *Calculating Devices*, *Hyperbolic Functions*, *Linear Programming*, *Statistics*, *Sets*, *Functions*, *Numerical Methods*, *Coordinate Geometry I*, *Integration* and *Differentiation*. Advanced Mathematics 2 consisted of *Probability*, *Logic*, *Vectors*, *Complex Numbers*, *Trigonometry*, *Algebra*, *Differential Equations* and *Coordinate Geometry II*.

The analysis of responses in each topic reveals that 14 out of 18 topics had good performance. The topics with good performance were *Logic (98.4%)*, *Coordinate Geometry II (92.8%)*, *Trigonometry (92.0%)*, *Sets (91.4%)*, *Linear Programming (87.8%)*, *Statistics (82.3%)*, *Algebra (81.9%)*, *Calculating Devices (79.8%)*, *Vectors*

(71.9%), Hyperbolic Functions (70.9%), Numerical Methods (66.9%), Differential Equations (64.6%), Complex Numbers (63.6%) and *Functions (60.4%)* as summarized in Appendix I.

The good performance in these topics was contributed by candidates' ability to:

(a) use a scientific calculator to perform computations of various mathematical expressions.
(b) solve related problems by using hyperbolic definitions.
(c) formulate mathematical models and solve linear programming problems graphically.
(d) apply the concepts of statistics to determine the measures of dispersion.
(e) simplify set expressions and solve related problems by using Venn diagrams.
(f) draw graphs of polar equations as well as rational, exponential and logarithmic functions.
(g) apply the trapezium rule and Secant method in solving related problems.
(h) find equations of tangent and normal to various conic sections as well as forming equations of circle in standard form.
(i) determine the probabilities by using the concept of normal distribution, conditional probability and fundamental principle of counting.
(j) test the validity of an argument by using the laws of algebra of propositions and truth tables.
(k) apply the vector concepts to evaluate position vectors, unit vectors and work done by forces.
(l) apply the concept of complex numbers in solving problems.
(m) use trigonometric knowledge and skills to simplify and solve trigonometric problems.
(n) apply the inverse matrix method in solving simultaneous linear equations involving three unknowns.
(o) apply proper methods and techniques of differential equations in solving real life problems.

In spite of good performance in most of the topics, three topics of *Probability, Coordinate Geometry I* and *Differentiation* had an average performance of 55.0, 54.1, and 36.2 per cent, respectively. However, the topic of *Integration* had a weak performance of 22.6 per cent. The weak performance was mainly contributed by candidates' inability to:

(a) apply the method of integration by parts in integrating the product of algebraic and exponential functions.
(b) use partial fractions to evaluate definite integrals.

The analysis of the candidates' performance shows that, fifteen (15) topics were performed well, one (1) topic had average performance and two (2) topics were poorly performed in 2021. Comparatively, the analysis of the candidates' performance per topic reveals that, there was a fall in performance in nine (9) topics in 2022 compared to that of 2021 as indicated in Appendix II.

4.0 CONCLUSION AND RECOMMENDATIONS

4.1 Conclusion

This report revealed the strengths and weaknesses of the candidates' responses to each question on the examined topics. Generally, the performance in 2022 was good; 97.89 per cent of the candidates passed the examination compared to 93.49 per cent who passed in 2021.

The analysis showed that the candidates had good performance in 14 topics, average performance in 3 topics and weak performance in 1 topic. The weak performance was contributed by failure of the candidates to apply the method of integration by parts in integrating the product of algebraic and trigonometric functions, using correctly the method of partial fractions to evaluate the definite integrals and apply the Maclaurin's series to obtain a series of logarithmic functions. It is expected that stakeholders will make use of the recommendations presented in this report to more enhance the future performance on Advanced Mathematics examinations.

4.2 Recommendations

In order to improve the candidates' performance in this subject especially on the topic about Integration, the following are recommended:

(a) All concepts of functions, trigonometric functions, algebra, hyperbolic functions, coordinate geometry and differentiation should be given more emphasis in teaching and thoroughly be assessed before teaching integration.

(b) Since some candidates demonstrated poor performance on integration, these candidates might not be aware of how useful the topic is in fields such as economics, engineering and sciences. Teachers should explain to students how important the topic is.

(c) The poor performance of the candidates on integration calls for participatory teaching and learning process that will promote the performance.

(d) Some candidates demonstrated lack of familiarity in the way they solved problems related to integration. It is recommended that students be involved in a lot of prior examinations, tests and other forms of assessment just to familiarize the candidates.

(e) Teachers should conduct regular assessments on every subtopic of integration so as identify areas of difficulties and assist students accordingly.

Appendix I

Analysis of Candidates' Performance on each Topic in the 2022 Advanced Mathematics Examination

S/N	Topic	Question Number	The Percentage of Candidates who Passed	Remarks
1.	Logic	2	98.4	Good
2.	Coordinate Geometry II	8	92.8	Good
3.	Trigonometry	5	92.0	Good
4.	Sets	5	91.4	Good
5.	Linear Programming	3	87.8	Good
6.	Statistics	4	82.3	Good
7.	Algebra	6	81.9	Good
8.	Calculating Devices	1	79.8	Good
9.	Vectors	3	71.9	Good
10.	Hyperbolic Functions	2	70.9	Good
11.	Numerical Methods	7	66.9	Good
12.	Differential Equations	7	64.6	Good
13.	Complex Numbers	4	63.6	Good
14.	Functions	6	60.4	Good
15.	Probability	1	55.0	Average
16.	Coordinate Geometry I	8	54.1	Average
17.	Differentiation	10	36.2	Average
18.	Integration	9	22.6	Weak

Appendix II

Analysis of Candidates' Performance in each Topic in the 2021 and 2022 Advanced Mathematics Examinations

S/N	Topic	2021			2022	
		Question Number	The Percentage of Candidates who Passed	Remarks	The Percentage of Candidates who Passed	Remarks
1.	Logic	2	79.4	Good	98.4	Good
2.	Coordinate Geometry II	8	88.8	Good	92.8	Good
3.	Trigonometry	5	82.2	Good	92.0	Good
4.	Sets	5	91.2	Good	91.4	Good
5.	Linear Programming	3	89.7	Good	87.8	Good
6.	Statistics	4	80.3	Good	82.3	Good
7.	Algebra	6	68.1	Good	81.9	Good
8.	Calculating Devices	1	70.7	Good	79.8	Good
9.	Vectors	3	43.3	Average	71.9	Good
10.	Hyperbolic Functions	2	86.8	Good	70.9	Good
11.	Numerical Methods	7	89.8	Good	66.9	Good
12.	Differential Equations	7	68.1	Good	64.6	Good
13.	Complex Numbers	4	82.4	Good	63.6	Good
14.	Functions	6	88.1	Good	60.4	Good
15.	Probability	1	62.1	Good	55.0	Average
16.	Coordinate Geometry I	8	65.7	Good	54.1	Average
17.	Differentiation	10	25.2	Weak	36.2	Average
18.	Integration	9	33.9	Weak	22.6	Weak